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Infrared properties of forced magnetohydrodynamic turbulence

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Abstract. The dynamical renormalisation group (RG) is implemented to study the large-scale properties of incompressible conducting fluid stirred by random forces and currents. In contrast with Navier–Stokes turbulence, invariance properties and dimensional constraints do not always prescribe the renormalisation of the couplings. In dimensions $d > d_c \simeq 2.8$, the system displays two non-trivial regimes: a kinetic regime where the renormalisation of the transport coefficients is due to the kinetic small scales, and a magnetic regime where it is due to the magnetic small scales. The results for the magnetic regime are not identical with predictions from the direct interaction approximation; this is due to vertex renormalisation of the Lorentz force. In dimensions $2 \leqslant d \leqslant d_c$, with sufficiently strong external currents, there is no stable fixed point: run away of the figurative point occurs, making the RG approach self-defeating. In two dimensions, with weak forces and currents, the absolute equilibrium results of Fyfe and Montgomery are recovered.

1. Introduction

Infrared properties of randomly stirred fluids are now amenable to the dynamical renormalisation group (RG) technique (Forster *et al* 1976, 1977, Fournier 1977, de Dominicis and Martin 1979, Pouquet *et al* 1978, Garnier *et al* 1981). This method is a truly perturbative procedure in contrast with all the available closures, ranging from dimensional analysis to truncated renormalised expansions. The prototype of such expansions is the direct interaction approximation (DIA) (Kraichnan 1959, Leslie 1973), obtained by replacing the renormalised vertex by the bare vertex in the Dyson equation (Martin *et al* 1973). Although this procedure may be crude, it gives useful insight in situations where the energetics may be described in terms of renormalised forcing and dissipation (Orszag 1977, Rose and Sulem 1978, Sulem *et al* 1979). This is asymptotically correct for the largest scales of forced hydrodynamic turbulence, because the invariance of the Navier–Stokes equations under Galilean transformation prevents renormalisation of the vertex. RG calculations then corroborate DIA predictions. Besides, when the forcing is not renormalised, the crossover and the anomalous dimension of viscosity are prescribed by dimensional constraints (Fournier and Frisch 1978, de Dominicis and Martin 1979); the RG is then needed only to predict universal numbers and logarithmic corrections.

The situation is not so simple for conducting fluids when both large-scale random forces and random driving currents are prescribed. The corresponding magnetohydrodynamic (MHD) equations involve two coupled fields (velocity and magnetic fields). There is then no reason to expect dimensional arguments to be correct and, as

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we shall see, quite often they are not. Besides, when the Lorentz force is relevant, different couplings can tend to zero simultaneously. Then, the scaling laws are not prescribed by the sole fixed point; the discussion of relevance requires an actual asymptotic expansion of the renormalised equations, written as usual in terms of rescaled variables. Coming back to the primitive variables, one obtains 'effective equations' where the coupling constants are wavenumber dependent. In the context of the renormalisation group, a precise meaning is thus given to the phenomenological concepts of eddy viscosity, eddy diffusivity, eddy noise, turbulent Lorentz force, ...

2. Reduced couplings

For our purpose, the equations of motion for an incompressible conducting fluid are conveniently written in the form

$$\partial v/\partial t + \lambda_0 \mathscr{P}(v \cdot \nabla v) = \nu_0 \nabla^2 v + \lambda_0' \mathscr{P}(b \cdot \nabla b) + f \tag{1a}$$

$$\partial b/\partial t + \lambda_0 \operatorname{curl}(v \times b) = \eta_0 \nabla^2 b + j \tag{1b}$$

where v and b are the velocity and magnetic fields respectively. $\mathscr{P}$ denotes the incompressibility projection operator, ν_0 the viscosity and η_0 the magnetic diffusivity; λ_0 and λ_0' are formal expansion parameters, eventually taken equal to unity. Equal expansion parameters are taken in front of the bilinear terms on the left-hand side of equations (1) because they remain proportional throughout the renormalisation (see § 3). f is a prescribed force and j a magnetic driving which may be expressed in terms of prescribed currents. They are assumed to be independent, zero-mean, Gaussian random functions; their Fourier transforms have correlations given by

$$\langle \hat{f}_m(\boldsymbol{k}, \omega) \hat{f}_n(\boldsymbol{k}', \omega')\rangle = 2D_0^{\mathrm{V}} (2\pi)^{d+1} \mathscr{F}^{\mathrm{V}}(k) P_{mn}(\boldsymbol{k}) \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega') \tag{2a}$$

$$\langle \hat{j}_m(\boldsymbol{k}, \omega) \hat{j}_n(\boldsymbol{k}', \omega')\rangle = 2D_0^{\mathrm{M}} (2\pi)^{d+1} \mathscr{F}^{\mathrm{M}}(k) P_{mn}(\boldsymbol{k}) \delta(\boldsymbol{k}+\boldsymbol{k}')\delta(\omega+\omega') \tag{2b}$$

where d is the space dimension and

$$P_{mn}(\boldsymbol{k}) = \delta_{mn} - k_m k_n / k^2 \tag{3}$$

is the incompressibility projection operator. D_0^{V} and D_0^{M} are positive and measure the intensity of the kinetic and magnetic energy injections at a reference scale Λ^{-1}. We assume that the injection spectra per wavenumber

$$F^{\mathrm{V}}(k) = S_d k^{d-1} \mathscr{F}^{\mathrm{V}}(k) \tag{4a}$$

$$F^{\mathrm{M}}(k) = S_d k^{d-1} \mathscr{F}^{\mathrm{M}}(k) \tag{4b}$$

follow power laws in the limit $k \to 0$

$$F^{\mathrm{V}}(k) \sim k^{-r} \qquad \text{with } r = -3 + \varepsilon^{\mathrm{V}} \leqslant 0 \tag{5a}$$

$$F^{\mathrm{M}}(k) \sim k^{-\rho} \qquad \text{with } \rho = -3 + \varepsilon^{\mathrm{M}} \leqslant 0. \tag{5b}$$

In equations (4), $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the d-dimensional unit sphere. No helicity (i.e. correlation between velocity and vorticity, or between magnetic field and potential vector) is assumed: helicity has a destabilising effect on the large scale magnetic field (Moffatt 1978), which probably leads to an inverse cascade (Pouquet *et al* 1976, Pouquet and Patterson 1978); this effect does not seem tractable by the RG (Pouquet *et al* 1978).

To estimate the relative importance of the nonlinear terms at a given scale ξ^{-1}, we rescale the problem using ξ^{-1} as unit length and the viscous time as unit time; the amplitudes of the velocity and magnetic fields in the free problem are taken as unities. Equations (1) become

$$\frac{\partial v}{\partial t}+G^{1/2}(\xi)\mathcal{P}(v\cdot\nabla v)=\nabla^2 v+\kappa_0^2\frac{X(\xi)}{G^{1/2}(\xi)}\mathcal{P}(b\cdot\nabla b)+f \tag{6a}$$

$$\frac{\partial b}{\partial t}+G^{1/2}(\xi)(v\cdot\nabla b-b\cdot\nabla v)=\kappa_0\nabla^2 b+j \tag{6b}$$

where the somewhat improperly denoted electromotive force, curl$(v\times b)$, has been rewritten as $(v\cdot\nabla b-b\cdot\nabla b)$. Three reduced parameters appear: the advective coupling constant

$$G(\xi)=\lambda_0^2 D_0^{\mathrm{V}}\nu_0^{-3}\xi^{-\varepsilon^{\mathrm{V}}}=G_0\xi^{-\varepsilon^{\mathrm{V}}} \tag{7a}$$

which in view of (6*a*) can be considered as the squared kinetic Reynolds number at scale ξ^{-1}; the inverse magnetic Prandtl number

$$\kappa_0=\eta_0\nu_0^{-1} \tag{7b}$$

and the magnetic coupling constant

$$X(\xi)=\lambda_0\lambda_0' D_0^{\mathrm{M}}\eta_0^{-2}\nu_0^{-1}\xi^{-\varepsilon^{\mathrm{M}}}=X_0\xi^{-\varepsilon^{\mathrm{M}}}. \tag{7c}$$

The large-scale behaviour of the advective and magnetic coupling constants suggests a competition between two possible crossovers and a detailed analysis is thus required.

3. Small-scale elimination and recursion relations

Starting from the equations written in Fourier space (figure 1) with an ultraviolet cut-off at wavenumber Λ (taken as unity), the first step of the RG procedure consists in eliminating the small scales: the modes $v^>(k,\omega)$ and $b^>(k,\omega)$, with $e^{-l}<k<1$, are calculated in terms of the modes $v^<(k,\omega)$ and $b^<(k,\omega)$, with $0<k<e^{-l}$, to second order in λ_0 and λ_0', and substituted in the equation for $v^<$ and $b^<$. Many new couplings are generated. When only the couplings which can be relevant in the limit $k\to 0$ are retained (Wilson and Kogut 1974, Ma 1973, 1976) the equations satisfied by $v^<$ and $b^<$ (figure 2) reduce to the primitive MHD equations with renormalised coefficients (henceforth referred to by the subscript I).

The viscosity is enhanced under the effect of kinetic and magnetic small scales, and becomes

$$\nu_{\mathrm{I}}=\nu_0\left(1+\alpha_{\mathrm{V}}^{\nu}(G_0)\frac{e^{\varepsilon^{\mathrm{V}}l}}{\varepsilon^{\mathrm{V}}}+\alpha_{\mathrm{M}}^{\nu}(X_0)\frac{e^{\varepsilon^{\mathrm{M}}l}-1}{\varepsilon^{\mathrm{M}}}\right). \tag{8a}$$

Here

$$\alpha_{\mathrm{V}}^{\nu}(G_0)=\frac{d^2-d-\varepsilon^{\mathrm{V}}}{2d(d+2)}\frac{S_d}{(2\pi)^d}G_0 \tag{8b}$$

and

$$\alpha_{\mathrm{M}}^{\nu}(G_0)=\frac{d^2+d-4+\varepsilon^{\mathrm{M}}}{2d(d+2)}\frac{S_d}{(2\pi)^d}X_0. \tag{8c}$$

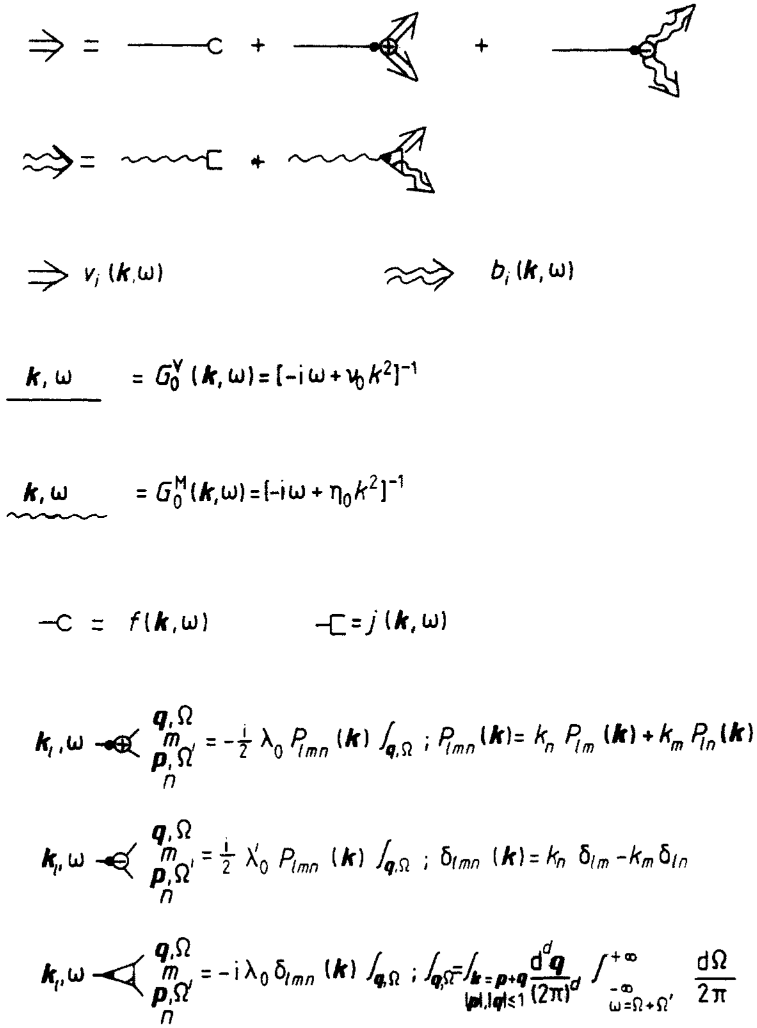


Figure 1. Diagrammatic representation of the MHD equations. The Fourier transform $u(\boldsymbol{k}, \omega)$ of a field $u(\boldsymbol{x}, t)$ is defined by $u(\boldsymbol{k}, t) = \int d\boldsymbol{x}\, d\omega\, u(\boldsymbol{x}, t) \exp[i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})]$.

In the renormalised magnetic diffusivity

$$\eta_1 = \eta_0\left(1 + \alpha_V^\eta(G_0, \kappa_0)\frac{e^{\varepsilon^V l} - 1}{\varepsilon^V} + \alpha_M^\eta(X_0, \kappa_0)\frac{e^{\varepsilon^M l} - 1}{\varepsilon^M}\right) \tag{9a}$$

the kinetic contribution

$$\alpha_V^\eta(G_0, \kappa_0) = \frac{d-1}{d}\frac{S_d}{(2\pi)^d}\frac{G_0}{\kappa_0(1+\kappa_0)} \tag{9b}$$

corresponds to a damping, while the magnetic contribution

$$\alpha_M^\eta(X_0, \kappa_0) = \frac{d-3}{d}\frac{S_d}{(2\pi)^d}\frac{X_0}{(1+\kappa_0)} \tag{9c}$$

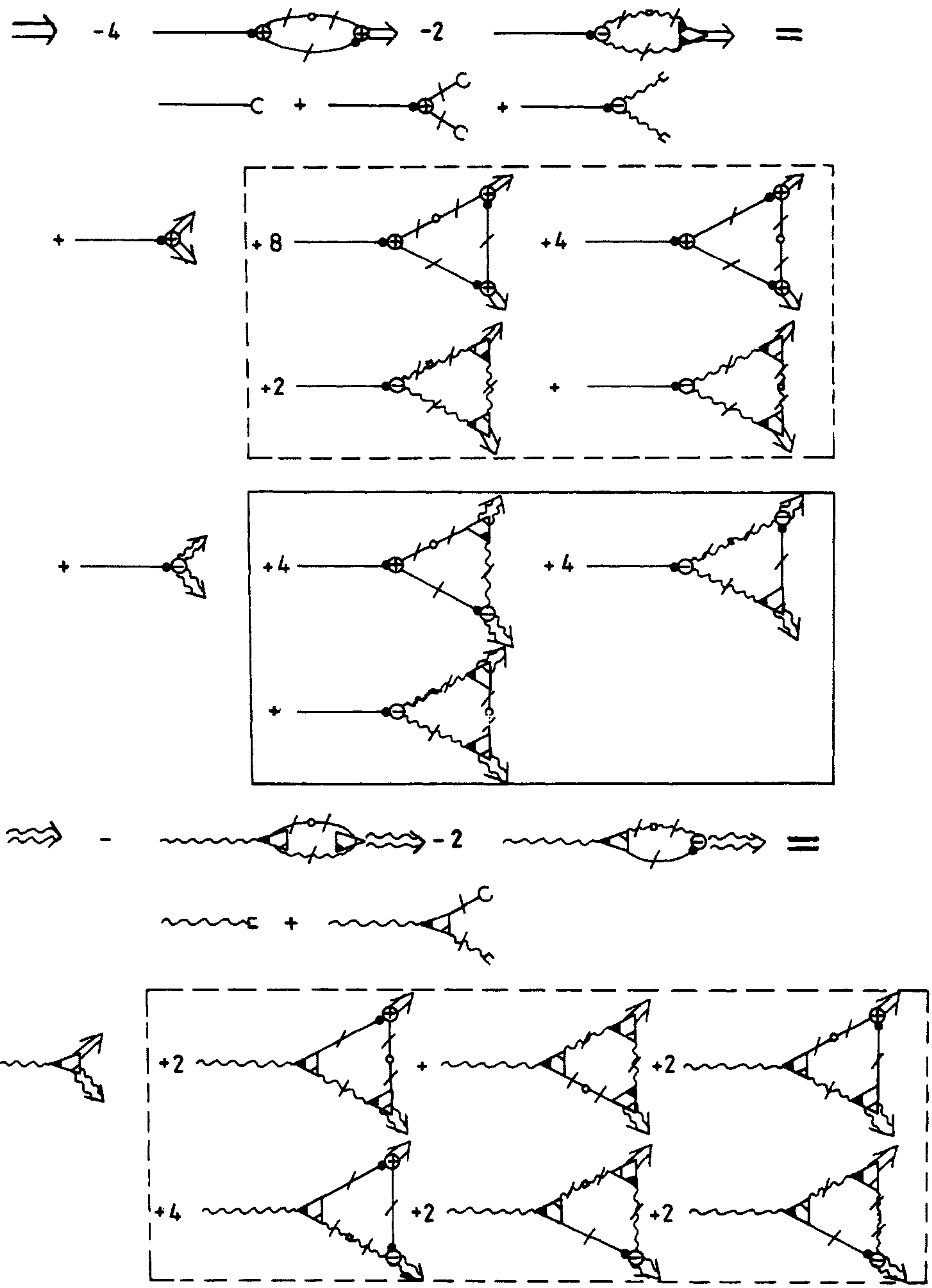


Figure 2. Equations for $v^{<}(\boldsymbol{k}, \omega)$ and $b^{<}(\boldsymbol{k}, \omega)$ with $|\boldsymbol{k}| < e^{-l}$. Dashed propagators indicate that the corresponding wavenumber belongs to the shell $e^{-l} \leqslant q \leqslant 1$. The correlation of the small scale forces and currents is represented by

$$ {}_{\boldsymbol{k},\omega}^{\;m}\!\!-\!\!\bigcirc\!\!-\!\!{}^{n}_{\boldsymbol{k}',\omega'} = \langle f_m(\boldsymbol{k}, \omega) f_n(\boldsymbol{k}', \omega') \rangle $$

$$ {}_{\boldsymbol{k},\omega}^{\;m}\!\!-\!\!\square\!\!-\!\!{}^{n}_{\boldsymbol{k}',\omega'} = \langle j_m(\boldsymbol{k}, \omega) j_n(\boldsymbol{k}', \omega') \rangle. $$

Only the possibly relevant terms in the limit $k \to 0$ have been retained; fluctuations of random operators have thus been neglected. Broken-line boxes contain the renormalisation of inertial force and electromotive force, which are found to be non-relevant; the solid-line box contains the renormalisation of the Lorentz force (which may be relevant).

is negative and thus destabilising in dimensions $d<3$; in two dimensions, this effect was already noted in the closure context (Pouquet 1978).

The source terms generated by the small scales in the equation for $v^<$ have correlation functions which behave like k^2 for $k\to 0$; the corresponding inputs per wavenumber are thus proportional to k^{d+1}. The relevance/irrelevance of these 'eddy noises' (Rose 1977) leads to the following classification:

models R^V (no renormalisation of the kinetic driving)

$$r>-(d+1) \qquad D_{\mathrm{I}}^{\mathrm{V}}=D_0^{\mathrm{V}}. \tag{10a}$$

models A^V (renormalisation of the kinetic driving)

$$r=-(d+1)$$

$$D_{\mathrm{I}}^{\mathrm{V}}=D_0\left(1+\alpha_{\mathrm{V}}^{D\mathrm{V}}(G_0)\frac{\mathrm{e}^{(2-d)l}-1}{2-d}+\alpha_{\mathrm{M}}^{D\mathrm{M}}(X_0,\kappa_0,G_0)\frac{\mathrm{e}^{(2\rho+4+d)l}-1}{2\rho+4+d}\right). \tag{10b}$$

Here

$$\alpha_{\mathrm{V}}^{D\mathrm{V}}(G_0)=\frac{d^2-2}{2d(d+2)}\frac{S_d}{(2\pi)^d}G_0 \tag{10c}$$

and

$$\alpha_{\mathrm{M}}^{D\mathrm{V}}(X_0,\kappa_0,G_0)=\frac{d^2-2}{2d(d+2)}\frac{S_d}{(2\pi)^d}\frac{X_0^2\kappa_0}{G_0}. \tag{10d}$$

models C^V: $r<-(d+1)$. The large scale properties of models $\mathrm{C^V}$ identify with those of model $\mathrm{A^V}$, as it appears when the small scale elimination is iterated (Forster *et al* 1977).

Similarly, the correlation of the eddy noise which arises in the $b^<$-equation behaves like $(d-2)k^2+\mathrm{O}(k^4)$. It is negative for $d<2$. This makes the continuation of the equations to dimensions $d<2$, already questioned for fluid turbulence (Frisch *et al* 1976), strongly problematic in MHD. The following classification results:

models R^M (no renormalisation of the magnetic driving)

$$\left.\begin{array}{ll} d>2 & \text{with } \rho>-(d+1) \\ \text{or} & \\ d=2 & \text{with } \rho>-5 \end{array}\right\} \quad D_{\mathrm{I}}^{\mathrm{M}}=D_0^{\mathrm{M}}. \tag{11a}$$

models A^M (renormalisation of the magnetic driving)

$$\left.\begin{array}{ll} d>2 & \text{with } \rho=-(d+1) \\ \text{or} & \\ d=2 & \text{with } \rho=-5 \end{array}\right\} \quad D_{\mathrm{I}}^{\mathrm{M}}=D_0^{\mathrm{M}}\left(1+\alpha_{\mathrm{V}}^{D\mathrm{M}}(G_0,\kappa_0)\frac{\mathrm{e}^{\varepsilon^{\mathrm{V}}l}-1}{\varepsilon^{\mathrm{V}}}\right) \tag{11b}$$

with for $d>2$ and $\rho=-(1+d)$

$$\alpha_{\mathrm{V}}^{D\mathrm{M}}(G_0,\kappa_0)=\frac{2(d-2)S_d}{d(2\pi)^d}\frac{G_0}{\kappa_0(1+\kappa_0)} \tag{11c}$$

and for $d=2$ and $\rho=-5$

$$\alpha_{\mathrm{V}}^{D\mathrm{M}}(G_0,\kappa_0)=\frac{1}{2}\frac{1}{2\pi}\frac{G_0}{\kappa_0(1+\kappa_0)}. \tag{11d}$$

Again the *models* C^M (magnetic driving weaker than in models A^M) are in the universality classes of the corresponding models A^M.

The renormalisation of the vertices arising in the left-hand side of equations (1) is negligible when compared with the bare vertices (see figure 2), which thus remain unchanged:

$$\lambda_{\mathrm{I}}=\lambda_0. \tag{12}$$

The elimination of the small scales preserves indeed both the Galilean invariance (Forster *et al* 1976, de Dominicis and Martin 1979), and the expression of the electromotive force. In contrast, the vertex associated to the Lorentz force is renormalised by two contributions with opposite signs:

$$\lambda_{\mathrm{I}}'=\lambda_0'\left(1+\alpha_{\mathrm{V}}^{\lambda'}(G_0,\kappa_0)\frac{\mathrm{e}^{\varepsilon^{\mathrm{V}}l}-1}{\varepsilon^{\mathrm{V}}}+\alpha_{\mathrm{M}}^{\lambda'}(X_0)\frac{\mathrm{e}^{\varepsilon^{\mathrm{M}}l}-1}{\varepsilon^{\mathrm{M}}}\right) \tag{13a}$$

wherein

$$\alpha_{\mathrm{V}}^{\lambda'}(G_0,\kappa_0)=-\frac{2}{d(d+2)}\frac{S_d}{(2\pi)^d}\frac{G_0}{\kappa_0} \tag{13b}$$

and

$$\alpha_{\mathrm{M}}^{\lambda'}(X_0)=\frac{2}{d(d+2)}\frac{S_d}{(2\pi)^d}X_0. \tag{13c}$$

The second step of the RG procedure consists in rescaling wavenumbers, frequencies and fields:

$$\tilde{k}=k\,\mathrm{e}^l \qquad \tilde{\omega}=\omega\exp\int_0^l z(l')\,\mathrm{d}l'$$

$$v^<(k,\omega)=\tilde{v}(\tilde{k},\tilde{\omega})\exp\int_0^l \tau(l')\,\mathrm{d}l' \qquad b^<(k,\omega)=\tilde{b}(\tilde{k},\tilde{\omega})\exp\int_0^l \sigma(l')\,\mathrm{d}l'. \tag{14}$$

This induces scalings on the drivings

$$\begin{aligned}\tilde{f}(\tilde{k},\tilde{\omega})&=f^<(k,\omega)\exp\int_0^l (z(l')-\tau(l'))\,\mathrm{d}l'\\ \tilde{j}(\tilde{k},\tilde{\omega})&=j^<(k,\omega)\exp\int_0^l (z(l')-\sigma(l'))\,\mathrm{d}l'\end{aligned} \tag{15}$$

and on the couplings

$$\nu(l)=\nu_{\mathrm{I}}\exp\int_0^l (z(l')-2)\,\mathrm{d}l'$$

$$\eta(l)=\eta_{\mathrm{I}}\exp\int_0^l (z(l')-2)\,\mathrm{d}l'$$

$$D^{\mathrm{V}}(l)=D_{\mathrm{I}}^{\mathrm{V}} \exp \int_0^l [3z(l')-2\tau(l')+(r+d+1)+(d-2)]\,\mathrm{d}l'$$

$$D_{\mathrm{M}}(l)=D_{\mathrm{I}}^{\mathrm{M}} \exp \int_0^l [3z(l')-2\sigma(l')+(\rho+d+1)+(d-2)]\,\mathrm{d}l'$$

$$\lambda(l)=\lambda_{\mathrm{I}} \exp \int_0^l [\tau(l')-(d+1)]\,\mathrm{d}l' \tag{16}$$

$$\lambda'(l)=\lambda_{\mathrm{I}}' \exp \int_0^l [2\sigma(l')-\tau(l')-(d+1)]\,\mathrm{d}l'.$$

Combining (8)–(16) and replacing l by an infinitesimal parameter, we obtain differential recursion relations for the running coupling constants:

models $\mathrm{R^V}$ $$\mathrm{d}D^{\mathrm{V}}/\mathrm{d}l=D^{\mathrm{V}}[3z-2+\varepsilon^{\mathrm{V}}+2(d-2)] \tag{17a}$$

models $\mathrm{A^V}$ $$\mathrm{d}D^{\mathrm{V}}/\mathrm{d}l=D^{\mathrm{V}}[3z-2+(d-2)+\alpha_{\mathrm{V}}^{D^{\mathrm{V}}}(G)+\alpha_{\mathrm{M}}^{D^{\mathrm{V}}}(X,\kappa,G)] \tag{17b}$$

models $\mathrm{R^M}$ $$\mathrm{d}D^{\mathrm{M}}/\mathrm{d}l=D^{\mathrm{M}}[3z-2\sigma+\varepsilon^{\mathrm{M}}+2(d-2)] \tag{17c}$$

models $\mathrm{A^M}$ $$\mathrm{d}D^{\mathrm{M}}/\mathrm{d}l=D^{\mathrm{M}}[3z-2\sigma+d-2+\alpha_{\mathrm{V}}^{D^{\mathrm{M}}}(G,\kappa)] \qquad \text{for } d>2 \tag{17d}$$

$$\mathrm{d}D^{\mathrm{M}}/\mathrm{d}l=D^{\mathrm{M}}[3z-2\sigma-2+\alpha_{\mathrm{V}}^{D^{\mathrm{M}}}(G,\kappa)] \qquad \text{for } d=2 \tag{17e}$$

and for all the models

$$\mathrm{d}\nu/\mathrm{d}l=\nu[z-2+\alpha_{\mathrm{V}}^{\nu}(G)+\alpha_{\mathrm{M}}^{\nu}(X)] \tag{17f}$$

$$\mathrm{d}\eta/\mathrm{d}l=\eta[z-2+\alpha_{\mathrm{V}}^{\eta}(G,\kappa)+\alpha_{\mathrm{M}}^{\eta}(X)] \tag{17g}$$

$$\mathrm{d}\lambda/\mathrm{d}l=\lambda[\tau-(d+1)] \tag{17h}$$

$$\mathrm{d}\lambda'/\mathrm{d}l=\lambda'[2\sigma-\tau-(d+1)+\alpha_{\mathrm{V}}^{\lambda'}(G,\kappa)+\alpha_{\mathrm{M}}^{\lambda'}(X)]. \tag{17i}$$

The anomalous dimensions α are expressed in terms of the reduced parameters (equations (8)–(13)) which at the lowest order satisfy:

models $\mathrm{R^V}$ $$\mathrm{d}G/\mathrm{d}l=G[\varepsilon^{\mathrm{V}}-3\alpha_{\mathrm{V}}^{\nu}(G)-3\alpha_{\mathrm{M}}^{\nu}(X)] \tag{18a}$$

models $\mathrm{A^V}$ $$\mathrm{d}G/\mathrm{d}l=G[(2-d)+\alpha_{\mathrm{V}}^{D^{\mathrm{V}}}(G)+\alpha_{\mathrm{M}}^{D^{\mathrm{V}}}(X,\kappa,G)-3\alpha_{\mathrm{V}}^{\nu}(G)-3\alpha_{\mathrm{M}}^{\nu}(X)] \tag{18b}$$

models $\mathrm{R^M}$ $$\mathrm{d}X/\mathrm{d}l=X[\varepsilon^{\mathrm{M}}+\alpha_{\mathrm{M}}^{\lambda'}(X)+\alpha_{\mathrm{V}}^{\lambda'}(G,\kappa)-\alpha_{\mathrm{V}}^{\nu}(G)-\alpha_{\mathrm{M}}^{\nu}(X)-2\alpha_{\mathrm{V}}^{\eta}(G,\kappa)-2\alpha_{\mathrm{M}}^{\eta}(X,\kappa)] \tag{18c}$$

models $\mathrm{A^M}$

$d>2$ $$\mathrm{d}X/\mathrm{d}l=X[(2-d)+\alpha_{\mathrm{V}}^{D^{\mathrm{M}}}(G,\kappa)+\alpha_{\mathrm{M}}^{\lambda'}(X)+\alpha_{\mathrm{V}}^{\lambda'}(G,\kappa)-\alpha_{\mathrm{V}}^{\nu}(G)-\alpha_{\mathrm{M}}^{\nu}(X)-2\alpha_{\mathrm{V}}^{\eta}(G,\kappa)-2\alpha_{\mathrm{M}}^{\eta}(X,\kappa)] \tag{18d}$$

$d=2$ $$\mathrm{d}X/\mathrm{d}l=X[-2+\alpha_{\mathrm{V}}^{D^{\mathrm{M}}}(G,\kappa)+\alpha_{\mathrm{M}}^{\lambda'}(X)+\alpha_{\mathrm{V}}^{\lambda'}(G,\kappa)-\alpha_{\mathrm{V}}^{\nu}(G)-2\alpha_{\mathrm{V}}^{\eta}(G,\kappa)-2\alpha_{\mathrm{M}}^{\eta}(X,\kappa)]. \tag{18e}$$

Besides, for all the models

$$\mathrm{d}\kappa/\mathrm{d}l=\kappa[\alpha_{\mathrm{V}}^{\eta}(G,\kappa)+\alpha_{\mathrm{M}}^{\eta}(G,\kappa)-\alpha_{\mathrm{V}}^{\nu}(G)-\alpha_{\mathrm{M}}^{\nu}(X)]. \tag{18f}$$

4. Trivial and kinetic regimes

A naive analysis neglecting renormalisation suggests that, if the magnetic energy is negligible when compared with the kinetic energy, the Lorentz force is irrelevant. The magnetic field then behaves as a passive vector, advected by turbulence.

When the Lorentz force is dropped out (a constraint preserved by renormalisation), two reduced parameters remain: the inverse Prandtl number κ, which for $d \geq 2$ and $\varepsilon^{\mathrm{V}} \geq (2-d)$ obeys

$$\frac{\mathrm{d}\kappa}{\mathrm{d}l} = -G\frac{S_d}{(2\pi)^d}\frac{1}{1+\kappa}\left(\kappa(1+\kappa)\frac{d^2-d-\varepsilon^{\mathrm{V}}}{2d(d+2)} - \frac{d-1}{d}\right) \tag{19}$$

and the advective coupling constant G which satisfies a model-dependent equation.

In the *models* R^{V} $(d \geq 2;\ \varepsilon^{\mathrm{V}} > 2-d)$

$$\mathrm{d}G/\mathrm{d}l = G[\varepsilon^{\mathrm{V}} - 3\alpha_{\mathrm{V}}^{\nu}(G)]. \tag{20}$$

The crossover parameter ε^{V} governs the topology of the flow (see figure 3):

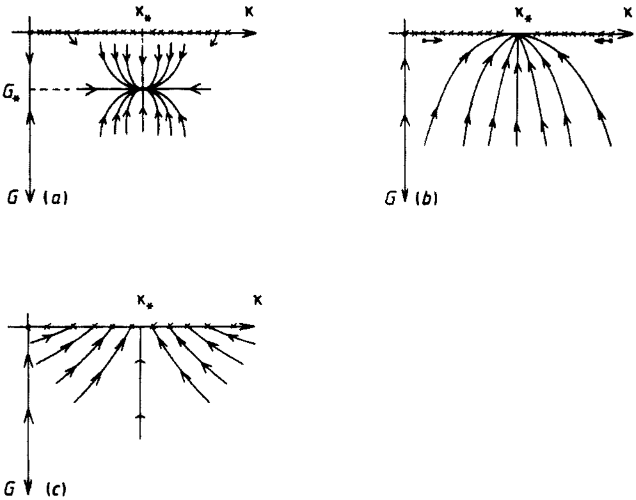


Figure 3. Flow diagrams of the reduced parameters in the passive regimes. G is the square Reynolds number and κ the inverse magnetic Prandtl number. The magnetic driving does not affect the flow diagrams. These flows also remain qualitatively the same whether or not the external force is renormalised. (a) corresponds to the non-trivial side of the crossover ($\varepsilon^{\mathrm{V}} > 0$ in models R^{V}) (kinetic regime), (b) to the marginal case ($\varepsilon^{\mathrm{V}} = 0$ in models R^{V} or $d = 2$ in models A^{V}); in both cases a universal Prandtl number is obtained; to the lowest order in the crossover parameter, its inverse reads $\kappa_* = \frac{1}{2}(-1 + (1 + 8(d+2)/d)^{1/2})$; ($c$) in the trivial case ($\varepsilon^{\mathrm{V}} < 0$ in models R^{V}; $d > 2$ in models A^{V}), universality is broken.

(i) For (small) $\varepsilon^{\mathrm{V}} > 0$, G is exponentially driven to

$$G_* = \frac{(2\pi)^d}{S_d}\frac{2(d+2)}{3(d-1)}\varepsilon^{\mathrm{V}} \tag{21}$$

and the anomalous dimension of viscosity is given by

$$\alpha_{\mathrm{V}}^{\nu}(G_*) = \tfrac{1}{3}\varepsilon^{\mathrm{V}}. \tag{22}$$

The inverse Prandtl number κ has a universal limit which, at the lowest order in ε^{V}, reads

$$\kappa_*(d) = \tfrac{1}{2}[-1 + (1 + 8(d+2)/d)^{1/2}]. \tag{23}$$

(ii) For $\varepsilon^{V} = 0$,

$$G(l) \sim \frac{2(d+2)}{3(d-1)} \frac{(2\pi)^d}{S_d} \frac{1}{l} \tag{24}$$

$$\alpha_V^{\nu}(G(l)) \sim 1/3l \tag{25}$$

and κ is still driven to κ_*; universality is thus preserved in the marginal case.

(iii) For $d > 2$ and $2-d < \varepsilon^{V} < 0$, G goes exponentially to zero; the nonlinear effects are negligible in the infrared limit, and the whole line $G = 0$ is stable; this degeneracy makes the universality break down: the limiting Prandtl number depends on the bare parameters (see figure 3*c*).

In *models* A^V, the kinetic modes are in thermal equilibrium, and a fluctuation dissipation theorem holds for the velocity field. This implies equality of the anomalous dimensions of noise $\alpha_V^{D^V}$ and of dissipation α_V^{ν}. For $d \geq 2$ and $\varepsilon^{V} = 2-d$, equation (18*b*) thus reads:

$$\mathrm{d}G/\mathrm{d}l = G[(2-d) - 2\alpha_V^{\nu}(G)]. \tag{26}$$

The flows are qualitatively similar to those of the model R^V, with $(2-d)$ for crossover parameter:

(i) For $d = 2$, when $l \to \infty$

$$G(l) \sim 8\pi/l \tag{27}$$

$$\alpha_V^{\nu}(l) \sim 1/2l \tag{28}$$

and κ has the same limit as in model R^V (see figure 3*b*):

$$\kappa_*(d=2) = \tfrac{1}{2}(-1 + 17^{1/2}). \tag{29}$$

The same universal Prandtl number was obtained by Forster *et al* (1977) for a passive scalar advected by a two-dimensional flow at thermal equilibrium, with deterministic initial concentration and no source of passive scalar.

(ii) For $d > 2$, the situation is analogous to that described in (iii) for the model R^V (see figure 3*c*).

Returning to the parameters ν, η, D^V, D^M and λ, we choose the scaling factors z, τ and σ defined in equations (14), in such a way that ν, D^V and D^M remain fixed. This ensures that $\lambda(l) = G^{1/2}(l)(\nu^3/D^V)^{1/2}$ goes to zero, or to a value $O(\sqrt{\varepsilon^V})$, and that $\eta(l) = \kappa(l)\nu$ has a finite limit $\eta(\infty)$. When $l \to \infty$, the nonlinear terms of the equations satisfied by the rescaled fields thus vanish or are of order $O(\sqrt{\varepsilon^V})$ when evaluated at wavenumber $k = O(1)$ and compared with the linear terms. They are thus negligible at the lowest order in ε. Coming back to the original variables, we obtain effective equations for velocity and magnetic fields at wavenumber $k = e^{-l}$ and frequency ω:

$$[-i\omega + \nu(k)k^2]v(k, \omega) = f(k, \omega) \tag{30a}$$

$$[-i\omega + \eta(k)k^2]b(k, \omega) = j(k, \omega) \tag{30b}$$

where f and j are white noise in time which satisfy

$$\langle |f(k,\omega)|^2\rangle \propto D^{\mathrm{V}}(k)k^{4-d-\varepsilon^{\mathrm{V}}}$$
$$\langle |j(k,\omega)|^2\rangle \propto D^{\mathrm{M}}(k)k^{4-d-\varepsilon^{\mathrm{M}}}. \tag{30c}$$

In the trivial cases (model R^{V} with $\varepsilon^{\mathrm{V}}<0$, or model A^{V} with $d>2$),

$$\nu(k)\sim\eta(k)\propto k^0 \tag{31}$$

and the characteristic frequencies scale like k^2. In the marginal cases, log corrections occur:

models R^{V} $\quad d>2, \varepsilon^{\mathrm{V}}=0 \quad \nu(k)\sim\eta(k)\propto(\log 1/k)^{1/3}$ (32a)

models A^{V} $\quad d=2, \varepsilon^{\mathrm{V}}=0 \quad \nu(k)\sim\eta(k)\propto(\log 1/k)^{1/2}$. (32$b$)

In the non-trivial case (models R^{V} with $\varepsilon^{\mathrm{V}}>0$) the renormalised transport coefficients scale as

$$\nu(k)\sim\eta(k)\sim k^{-\varepsilon^{\mathrm{V}}/3} \tag{33}$$

with exponents prescribed by the kinetic forcing only; we shall refer to this regime as the 'kinetic regime'.

For all the passive regimes, the long-time behaviour of the correlation functions is thus independent of the magnetic driving. However, the spatial correlations of the magnetic field depend on both ε^{V} and ε^{M}. Table 1 gives the kinetic and magnetic energy spectra $E^{\mathrm{V}}(k)$ and $E^{\mathrm{M}}(k)$, obtained by integration of $\langle|v(\boldsymbol{k},\omega)|^2\rangle$ and $\langle|b(\boldsymbol{k},\omega)|^2\rangle$ over the frequency ω and over the sphere of radius $|\boldsymbol{k}|$. (See also figures 5.)

Model $R^{V}R^{M}$: $\varepsilon^{\mathrm{V}}>2-d$ and $\{\varepsilon^{\mathrm{M}}>-2$ if $d=2\}$; $\{\varepsilon^{\mathrm{M}}>(2-d)$ if $d>2\}$.

The kinetic and magnetic energy spectra result from a balance between external drivings and renormalised dissipation or diffusion.

Model $A^{V}R^{M}$: $\varepsilon^{\mathrm{V}}=2-d$ and $\{\varepsilon^{\mathrm{M}}>-2$ if $d=2\}$; $\{\varepsilon^{\mathrm{M}}>(2-d)$ if $d>2\}$.

The anomalous dimensions of force intensity and viscosity are equal. Detailed balance holds and the kinetic energy is equally distributed among the Fourier modes; in contrast the magnetic spectrum results from an equilibrium between (external) current and (renormalised) diffusion. A log dependence is obtained in dimension $d=2$.

Model $A^{V}A^{M}$ for $d>2$: $\varepsilon^{\mathrm{V}}=2-d$; $\varepsilon^{\mathrm{M}}=(2-d)$.

All the possible anomalous dimensions are exponentially driven to zero; the energy spectra correspond to equipartition of both kinetic and magnetic energies.

Model $A^{V}A^{M}$ for $d=2$: $\varepsilon^{\mathrm{V}}=0$, $\varepsilon^{\mathrm{M}}=-2$.

The statistics of the velocity field is the same as in model $\mathrm{A}^{\mathrm{V}}\mathrm{R}^{\mathrm{M}}$. In the equation for the magnetic field, the anomalous dimensions of the diffusion, $\alpha_{\mathrm{V}}^{\eta}$, and of the current intensity, $\alpha_{\mathrm{V}}^{D^{\mathrm{M}}}$, are equal (see equations (9a) and (11d) with $d=2$); the log correction of the model $\mathrm{A}^{\mathrm{V}}\mathrm{R}^{\mathrm{M}}$ thus disappears. The magnetic spectrum is proportional to k^3, corresponding to an equipartition of the mean squared potential vector a ($b=\nabla\times a$) which is an invariant of the two-dimensional inviscid MHD equations. This is consistent with the absolute equilibrium distributions derived by Fyfe and Montgomery (1976) from an ultraviolet truncated version of the MHD equations with no force and no dissipation.

Table 1. Infrared scaling of the kinetic and magnetic energy spectra $E^{\mathrm{V}}(k)$ and $E^{\mathrm{M}}(k)$ for the passive vector regimes. The symbol * indicates ranges of parameters in which these regimes are unstable relatively to the Lorentz coupling. Note that the scaling behaviour of the kinetic energy spectrum is independent of ε^{M} (magnetic driving). See also figure 5 where the spectral exponents are plotted as functions of ε^{V} and ε^{M} for the passive and magnetic regimes.

$d=2$

		$\mathrm{A^V; C^V}$ $r\leqslant -3$	$\mathrm{R^V}$ $r>-3$
	ε^{M} \ ε^{V}	$\varepsilon^{\mathrm{V}}\leqslant 0$	$\varepsilon^{\mathrm{V}}>0$, small
$\mathrm{R^M}$ $\rho>-5$	$0<\varepsilon^{\mathrm{M}}<a_{\mathrm{cr}}\lvert\varepsilon^{\mathrm{V}}\rvert$	*	$E^{\mathrm{M}}(k)\propto k^{1-(a-1/3)\varepsilon^{\mathrm{V}}}$
	$-2<\varepsilon^{\mathrm{M}}<0$	$E^{\mathrm{M}}(k)\propto k^{1-\varepsilon^{\mathrm{M}}}[\log k^{-1}]^{-1/2}$ $E^{\mathrm{V}}(k)\propto k$	$E^{\mathrm{V}}(k)\propto k^{1-(2/3)\varepsilon^{\mathrm{V}}}$
$\mathrm{A^M; C^M}$ $\rho\leqslant -5$	$\varepsilon^{\mathrm{M}}\leqslant -2$	$E^{\mathrm{M}}(k)\propto k^3$	$E^{\mathrm{M}}(k)\propto k^3$

$d>2$

		$\mathrm{A^V: C^V}$ $r\leqslant -(d+1)$	$\mathrm{R^V}$ $r>-(d+1)$		
	ε^{M} \ ε^{V}	$\varepsilon^{\mathrm{V}}\leqslant(2-d)$	$(2-d)<\varepsilon^{\mathrm{V}}<0$	$\varepsilon^{\mathrm{V}}=0$	$\varepsilon^{\mathrm{V}}>0$, small
$\mathrm{R^M}$ $\rho>-(d+1)$	$0<\varepsilon^{\mathrm{M}}<a_{\mathrm{cr}}\lvert\varepsilon^{\mathrm{V}}\rvert$	*	*	*	$E^{\mathrm{M}}(k)\propto k^{1-(a-1/3)\varepsilon^{\mathrm{V}}}$
	$(2-d)<\varepsilon^{\mathrm{M}}<0$	$E^{\mathrm{M}}(k)\propto k^{1-\varepsilon^{\mathrm{M}}}$ $E^{\mathrm{V}}(k)\propto k^{d-1}$	$E^{\mathrm{M}}(k)\propto k^{1-\varepsilon^{\mathrm{M}}}$ $E^{\mathrm{V}}(k)\propto k^{1-\varepsilon^{\mathrm{V}}}$	$E^{\mathrm{M}}(k)\propto k^{1-\varepsilon^{\mathrm{M}}}[\log k^{-1}]^{-1/3}$ $E^{\mathrm{V}}(k)\propto k[\log k^{-1}]^{-1/3}$	$E^{\mathrm{V}}(k)\propto k^{1-(2/3)\varepsilon^{\mathrm{V}}}$
$\mathrm{A^M; C^M}$	$\varepsilon^{\mathrm{M}}\leqslant(2-d)$	$E^{\mathrm{M}}(k)\propto k^{d-1}$	$E^{\mathrm{M}}(k)\propto k^{d-1}$	$E^{\mathrm{M}}(k)\propto k^{d-1}[\log k^{-1}]^{-(d-3)/3(d-1)}$	$E^{\mathrm{M}}(k)\propto k^{d-1-(d-3)\varepsilon^{\mathrm{V}}/3(d-1)}$

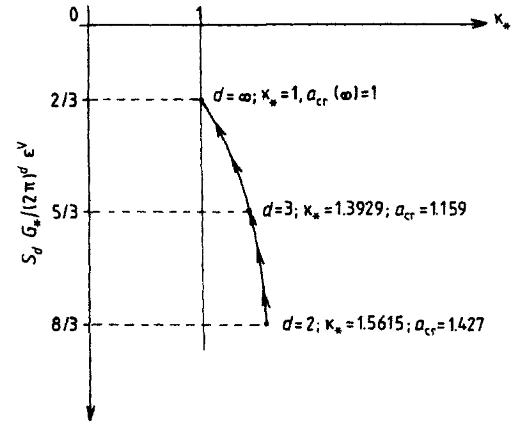


Figure 4. Trajectory of the fixed point in the (G, κ) plane for the kinetic regime of model R^VR^M when the space dimension d varies. ε^V is fixed to a small positive value. To include the limit $d\to\infty$, the reduced coupling G is scaled by a factor $S_d/(2\pi)^d\varepsilon^V$. The fixed points are labelled by the critical value $a_{cr}(d)$ of $a=\varepsilon^M/|\varepsilon^V|$ for which they become unstable in the transverse direction.

Model R^VA^M: $\varepsilon^V > 2-d$, and $\{\varepsilon^M=-2$ if $d=2\}$; $\{\varepsilon^M = 2-d$ if $d>2\}$.

The kinetic spectrum is the same as in the model R^VR^M. In dimension two, the magnetic spectrum corresponds to an equipartition of mean squared potential vector. In dimension $d>2$ for weak kinetic forcing ($\varepsilon^V<0$), equipartition of magnetic energy holds; but for $\varepsilon^V>0$, the small kinetic scales generate a relevant 'magnetic noise', without an *a priori* relation with the 'turbulent diffusion'; the magnetic spectrum then depends on the precise expression of the anomalous dimensions. Note that the correction to equipartition disappears in dimension 3, where renormalised diffusion and magnetic noise have the same scaling behaviour.

Let us now investigate the persistence of the above scaling and universality results when the Lorentz coupling is restored. When linearised around one of the fixed points $(G_F, \kappa_F, X_F=0)$ stable in the passive vector problem, the recursion relations read

$$\frac{\mathrm{d}}{\mathrm{d}l}\begin{bmatrix} G-G_F \\ \kappa-\kappa_F \\ X \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 & p \\ r & \mu_2 & q \\ 0 & 0 & \mu_3 \end{bmatrix}\begin{bmatrix} G-G_F \\ \kappa-\kappa_F \\ X \end{bmatrix} \tag{34}$$

where μ_1 and μ_2 are negative or zero. The stability is thus governed by the sign of μ_3.

Models A^M. For $d=2$, the magnetic coupling X is driven to zero (see equation (18*c*)). For $d>2$, equation (18*d*) thus yields

$$\mu_3=(2-d)+\alpha_V^{DM}(G_F,\kappa_F)+\alpha_V^{\lambda'}(G_F,\kappa_F)-\alpha_V^{\nu}(G_F)-2\alpha_V^{\eta}(G_F,\kappa_F).$$

So, when $\varepsilon^V\leqslant 0$, G_F is zero and $\mu_3=2-d$ is negative. When $\varepsilon^V>0$,

$$\alpha_V^{\nu}(G_*)=\alpha_V^{\eta}(G_*,\kappa_*)=\varepsilon^V/3$$

and

$$\mu_3=2-d-\frac{2}{d(d+2)}\frac{S_d}{(2\pi)^d}\frac{G_*}{\kappa_*}-\varepsilon^V\frac{d-4/3}{d-1}$$

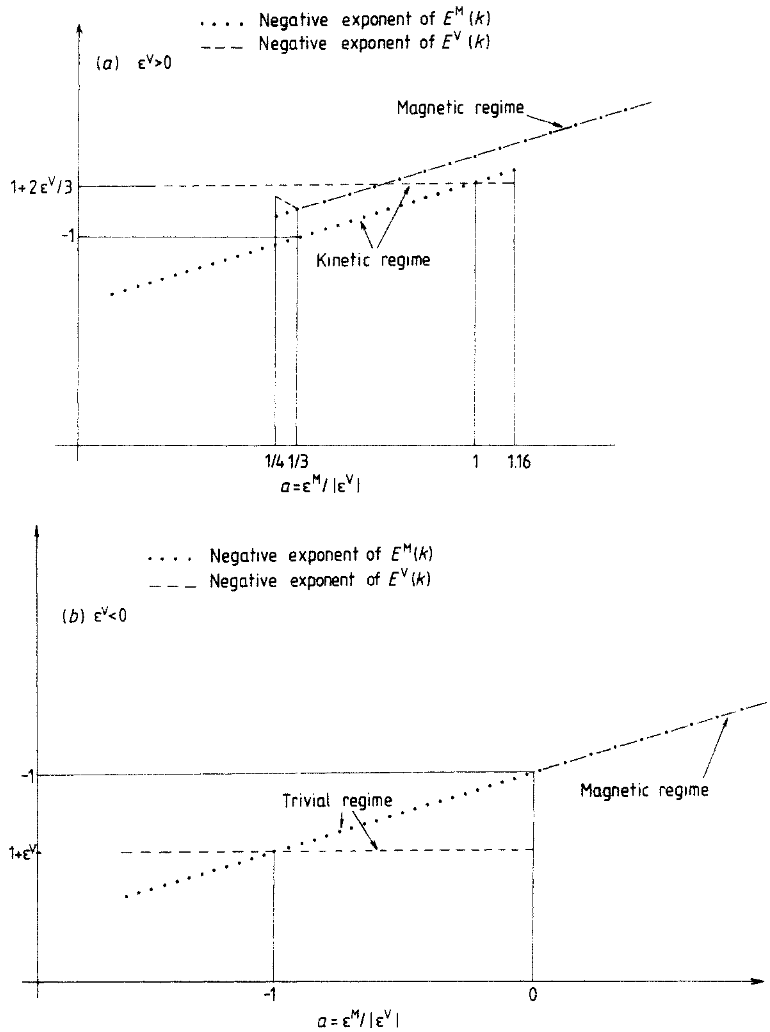


Figure 5. Negative exponents of kinetic and magnetic energy spectra for $k\to 0$, in model R^VR^M in dimension three, when $a=\varepsilon^M/|\varepsilon^V|$ varies. ε^V is fixed to a (small) positive value in (*a*) and to a negative value in (*b*). Note the possible coexistence of the kinetic and magnetic regimes, and also the fact that in the three regimes the dominant energy may be kinetic or magnetic. In the magnetic regime, when $E^V(k)$ and $E^M(k)$ have the same exponent, they satisfy $E^V(k)/E^M(k)\sim(\varepsilon^M)^2$.

is also negative. It follows that all the fixed points of the models A^M remain stable when the Lorentz force is reintroduced. As a consequence, the scaling relations and the universal values of the Prandtl number, obtained in the context of the passive vector, extend to the MHD problem.

Models R^M. Then

$$\mu_3=\varepsilon^M+\alpha_V^\lambda(G_F,\kappa_F)-\alpha_V^\nu(G_F)-2\alpha_V^\eta(G_F,\kappa_F).$$

When $G_F=0$, μ_3 reduces to ε^M. Consequently, in the case $\{\varepsilon^V<0,\ \varepsilon^M<0\}$, all the points of the line $G=0$ are stable and universality is not restored. For $\varepsilon^M>0$, these fixed points are unstable in the transverse direction. When G_F is positive,

$$\mu_3 = \varepsilon^M + \alpha_V^{\lambda'}(G_*, \kappa_*) - \varepsilon^V;$$

the stability is thus governed by the sign of $a - a_{cr}(d)$ where

$$a = \varepsilon^M / |\varepsilon^V| \tag{35}$$

$$a_{cr}(d) = 1 + (1+\kappa_*)2/3(d+2)(d-1) \tag{36}$$

and $\kappa_*(d)$ is given by equation (23). In the range $\{\varepsilon^V > 0; 1 < a < a_{cr}(d)\}$ the magnetic energy dominates the kinetic energy in the infrared limit while the Lorentz force is negligible. When d goes to infinity, this effect due to the renormalisation of the Lorentz force disappears: $a_{cr}(d) \to 1$ (see also figure 5(a)).

As already noted (Fournier and Frisch 1978, Sulem *et al* 1979), the eddy viscosity and the kinetic energy spectrum in the kinetic regime can readily be recovered using a dimensional argument of the Kolmogorov–Obukhov–Heisenberg type: the nonlinear interaction in the Navier–Stokes equation is modelised by a turbulent viscosity $\nu(k)$ expressed in terms of the local energy spectrum by

$$\nu(k) \sim (E^V(k)/k)^{1/2}. \tag{37}$$

The external injection $F(k) \sim k^{3-\varepsilon^V}$ permits the system to achieve a self-similar steady state where

$$\nu(k) \sim \left(\frac{1}{k}\frac{F^V(k)}{\nu(k)k^2}\right)^{1/2} \propto k^{-\Delta} \tag{38}$$

with an anomalous dimension

$$\Delta = \varepsilon^V/3. \tag{39}$$

Equation (37) remains valid in a situation of thermal equilibrium with an equipartition energy spectrum. For $d > 2$, it gives

$$\nu(k) \sim k^{(d-2)/2} \tag{40}$$

in agreement with the RG prediction (Forster *et al* 1977).

Note that the iteration to a fixed point, implemented in the RG scheme, plays a part similar to the bootstrapping involved in equation (37). The lack of renormalisation of the inertial terms may actually be obtained non-perturbatively in a field theoretical formulation of the RG (see e.g. de Dominicis and Peliti 1978). This is then, through a Ward identity, a consequence of the Galilean invariance (de Dominicis and Martin 1979). Hence, as long as the viscosity remains the unique renormalised coupling, its anomalous dimension is prescribed by dimensional constraints and need not be evaluated perturbatively (de Dominicis and Martin 1979).

These remarks partially extend to the passive regimes of MHD: there is a unique characteristic time, and eddy viscosity and diffusivity are proportional. The magnetic spectrum is thus easily calculated when the magnetic driving is not renormalised or when a fluctuation dissipation theorem holds. This excludes in particular the model R^VA^M with positive ε^V, where the renormalised magnetic driving has no simple relation with the eddy diffusivity. The exponent of the magnetic energy spectrum is thus dependent on the space dimension and its prediction requires a more refined analysis, such as resorting to turbulence closure. Since no vertex corrections are involved in the passive regimes, all the RG results will be correctly given by the DIA. The test field model (Kraichnan 1971, Sulem *et al* 1975), a Markovianised form of the DIA, will also give the correct result. Other Markovian closures such as the eddy damped quasi-normal Markovian approximation (Orszag 1977, Rose and Sulem 1978) will also work, except at the crossover where they produce improper log corrections. This is so because the latter closure involves an eddy damping rate which does not properly incorporate eddy viscosity effects from very small scales.

5. Magnetic regime

As seen in § 4, the trivial and kinetic regimes are unstable when the intensity of the external current is strong enough. Can we try to guess if a 'magnetic regime' with dominant Lorentz force is established? Such a regime will be associated to a fixed point with $G=0$ and X 'small'. Note that in dimension three, the magnetic contribution $\alpha_{\mathrm{M}}^{\eta}(X, \kappa)$ to the anomalous dimension of diffusivity vanishes, then leading to an infinite turbulent Prandtl number κ^{-1} (see equation (18f)). For dimension d less than three, the magnetic small scales tend to destabilise the magnetic large scales by a negative contribution to the eddy diffusivity. This suggests existence of a possible critical dimension under which the magnetic regime disappears. In this section, we discuss the possible magnetic regimes according to the space dimension and source intensities.

5.1. Dimension $d=3$

5.1.1. Model $R^{\mathrm{V}}R^{\mathrm{M}}$. Kinetic and magnetic drivings are not renormalised. This corresponds to ε^{V} and ε^{M} larger than (-1). Introducing $u=G/\kappa$, the recursion relations become

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}l} &= u\left[\varepsilon^{\mathrm{V}} - \frac{6-\varepsilon^{\mathrm{V}}}{30\pi^2}u\kappa - \frac{8+\varepsilon^{\mathrm{M}}}{30\pi^2}X - \frac{1}{3\pi^2}\frac{u}{1+\kappa}\right] \\
\frac{\mathrm{d}\kappa}{\mathrm{d}l} &= \kappa\left[\frac{1}{3\pi^2}\frac{u}{1+\kappa} - \frac{6-\varepsilon^{\mathrm{V}}}{60\pi^2}u\kappa - \frac{8+\varepsilon^{\mathrm{M}}}{60\pi^2}X\right] \qquad (41) \\
\frac{\mathrm{d}X}{\mathrm{d}l} &= X\left[\varepsilon^{\mathrm{M}} - \frac{4+\varepsilon^{\mathrm{M}}}{60\pi^2}X - \frac{6-\varepsilon^{\mathrm{V}}}{60\pi^2}u - \frac{1}{15\pi^2}\frac{\kappa+11}{\kappa+1}u\right].
\end{aligned}$$

For negative ε^{M}, the fixed point of the passive regime ($X=0$) is always reached. In contrast, for positive ε^{M} the system (41) has also a magnetic fixed point which at the lowest order in ε^{M} (here the expansion parameter) reads: ($u_*=0$, $\kappa_*=0$, $X_*=15\pi^2\varepsilon^{\mathrm{M}}$). When linearised around this magnetic fixed point, the recursion relations become

$$\frac{\mathrm{d}}{\mathrm{d}l}\begin{bmatrix} u \\ \kappa \\ X-X_* \end{bmatrix} = \begin{bmatrix} \varepsilon^{\mathrm{V}}-4\varepsilon^{\mathrm{M}} & 0 & 0 \\ 0 & -2\varepsilon^{\mathrm{M}} & 0 \\ -11\varepsilon^{\mathrm{M}} & 0 & -\varepsilon^{\mathrm{M}} \end{bmatrix}\begin{bmatrix} u \\ \kappa \\ X-X_* \end{bmatrix}. \qquad (42)$$

The stability thus requires $\varepsilon^{\mathrm{V}}<4\varepsilon^{\mathrm{M}}$, with the additional conditions $\varepsilon^{\mathrm{M}}>0$ (existence of the fixed point) and $\varepsilon^{\mathrm{V}}>-1$ (definition of the model R^{V}).

It is easily seen that the trivial, the kinetic and the magnetic fixed points are the only possibly stable fixed points of system (41). Note that the kinetic fixed point is stable for $\varepsilon^{\mathrm{M}}<0$, or $\varepsilon^{\mathrm{M}}>0$ with small $\varepsilon^{\mathrm{V}}>(1/1.16)\varepsilon^{\mathrm{M}}$. Thus for small positive ε^{V} and ε^{M}, satisfying $(1/1.16) < \varepsilon^{V}/\varepsilon^{M} < 4$, both fixed points are stable: universality is broken. Numerical integrations of the system (41) confirm that one or the other point can be reached, according to the values of the bare parameters (chosen of order ε for consistency). Note that if the renormalisation of the Lorentz force is artificially dismissed (as in the DIA), the overlapping of the two stability conditions is decreased but not suppressed: when $\alpha_{V}^{\lambda'}(G, \kappa)$ and $\alpha_{M}^{\lambda'}(X)$ are dropped in equation (41), the stability of the passive regime requires $\varepsilon^{V} > \varepsilon^{M}$, the magnetic fixed point becomes $X_{**} = (15\pi^2/2)\,\varepsilon^{M}$, and is stable for $\varepsilon^{V} < 2\varepsilon^{M}$.

To derive the asymptotic ($l \to \infty$) MHD equations satisfied by the rescaled fields in the magnetic regime, we choose the scaling factors such that the intensity of the magnetic driving and the viscosity remain fixed. The diffusivity $\eta(l) = \kappa(l)\nu_0$ then goes to zero. We choose the third scaling factor in such a way that $\lambda(l)$ tends to zero at the same rate as $\eta^2(l)$. As a consequence, the Lorentz parameter $\lambda'(l) = X(l)(\nu_0/D_0^{M})(\eta^2(l)/\lambda(l))$ has a limit λ'_* of order ε^{M} and the forcing intensity $D^{V}(l)$, which is proportional to $u(l)/\kappa^3(l)$, behaves like $\exp[(\varepsilon^{V} + 2\varepsilon^{M})l]$ near the fixed point (see equation (42)).

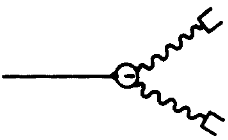
Let us first consider the case $\varepsilon^{V} < -2\varepsilon^{M}$. All the parameters remain bounded and their asymptotic behaviour suggests that in the equations for the rescaled fields, at wavenumber $\tilde{k} = O(1)$: the inertial force is negligible when compared with the viscous force; the electromotive force is negligible when compared with the magnetic diffusion; the external force is asymptotically negligible when compared with the Lorentz force which acts as a source term for the velocity field, with an intensity proportional to $(\varepsilon^{M})^2$.

Actually, several coupling parameters simultaneously go to zero, and among them the magnetic diffusivity. So the knowledge of the fixed point is not sufficient: the rate of convergence is also needed to discuss the relevance of the different terms. Furthermore, to be complete, this discussion should not be based on the behaviour of the sole coupling parameters but should also take into account the amplitude of the fields. This does not change the conclusion in the case $\varepsilon^{V} \leqslant -2\varepsilon^{M}$.

In the range $\varepsilon^{V} > -2\varepsilon^{M}$ we choose scaling factors which keep D^{M}, ν and D^{V} at finite values. All the other parameters then go to zero as follows

$$\begin{aligned} &\eta(l) \sim \exp(-2\varepsilon^{M} l) \\ &\lambda(l) \sim \exp[\tfrac{1}{2}(\varepsilon^{V} - 3\varepsilon^{M})l] \\ &\lambda'(l) \sim X_* \exp[-(\tfrac{1}{2}\varepsilon^{V} + \varepsilon^{M})l]. \end{aligned} \tag{43}$$

We shall check *a posteriori* that we can neglect the external, inertial and electromotive forces. We thus solve the resulting equations in the form (tildes are omitted):

$b^{\circ}(k, \omega; l) =$  $\qquad v^{\circ}(k, \omega; l) =$  (44)

where

 $= (-\mathrm{i}\omega + \eta(l)k^2)^{-1}$ $\qquad$ $= (-\mathrm{i}\omega + \nu_0 k^2)^{-1}$ (45)

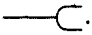
and the Lorentz vertex is proportional to $\lambda'(l)$. This yields

$$\begin{aligned} &\int \langle |b^{\circ}(k, \omega; l)|^2 \rangle \, \mathrm{d}\omega \sim \exp(2\varepsilon^{M} l) \\ &\int \langle |v^{\circ}(k, \omega; l)|^2 \rangle \, \mathrm{d}\omega \sim X_*^2 \exp[(3\varepsilon^{M} - \varepsilon^{V})l]. \end{aligned} \tag{46}$$

The amplitude and the characteristic time of the magnetic field thus increase indefinitely when $l \to \infty$. As a consequence the correlation of the Lorentz force becomes frequency dependent and the dynamics of the velocity field is not prescribed by the sole renormalised viscosity.

The above expansion is justified by comparing the effect on the energy spectra of the neglected and retained terms, all being calculated from v° and b°. Direct comparison of the different terms is not possible because of their frequency dependence. An example of calculation is given in the appendix. One finds that the contribution of the inertial force to the kinetic energy spectrum is proportional to $X_*^4 \exp(-\varepsilon^{\mathrm{V}} l)$, while the contribution of the Lorentz force is proportional to $X_*^2 \exp[(3\varepsilon^{\mathrm{M}} - \varepsilon^{\mathrm{V}})l]$; the contribution of the electromotive force to the magnetic energy spectrum behaves like $X_*^2 \exp(2\varepsilon^{\mathrm{M}} l)$. The approximations are thus consistent for $\varepsilon^{\mathrm{V}} < 3\varepsilon^{\mathrm{M}}$.

For stronger kinetic forcing ($3\varepsilon^{\mathrm{M}} < \varepsilon^{\mathrm{V}} < 4\varepsilon^{\mathrm{M}}$), the Lorentz force contribution to the kinetic energy spectrum is dominated by the contribution of the external force which is O(1). The velocity field then results from a balance between (renormalised) dissipation and external forcing:

$v^\circ(k, \omega; l) =$  . (47)

As before the contribution of the inertial force to the kinetic energy spectrum (proportional to $\exp[(-6\varepsilon^{\mathrm{M}} + \varepsilon^{\mathrm{V}})l]$) and that of the electromotive force to the magnetic spectrum (proportional to $\exp[(-2\varepsilon^{\mathrm{M}} + \varepsilon^{\mathrm{V}})l]$) are negligible.

The partition of the range of existence and stability of the magnetic fixed point into three subranges

$$\text{I: } -1 < \varepsilon^{\mathrm{V}} < -2\varepsilon^{\mathrm{M}} \qquad \text{II: } -2\varepsilon^{\mathrm{M}} \leq \varepsilon^{\mathrm{V}} < 3\varepsilon^{\mathrm{M}} \qquad \text{III: } 3\varepsilon^{\mathrm{M}} \leq \varepsilon^{\mathrm{V}} < 4\varepsilon^{\mathrm{M}}$$

was convenient for the discussion of relevance; it is however easily checked that the relevant terms and thus the effective equations for the original fields are the same in the ranges I and II. At wavenumber $k = \mathrm{e}^{-l}$ and frequency ω, these equations read

$$(-\mathrm{i}\omega + \nu(k)k^2)v(k, \omega) = \mathscr{L}(k)\widehat{\mathscr{P}(b \cdot \nabla b)}(k, \omega) \tag{48a}$$

$$(-\mathrm{i}\omega + \eta(k)k^2)b(k, \omega) = j(k, \omega) \tag{48b}$$

where $\hat{}$ denotes Fourier transform and

$$\begin{aligned} &\eta(k) \sim k^\circ \\ &\nu(k) \sim k^{-\Delta} && \Delta = \alpha_{\mathrm{M}}^{\nu}(X_*) = 2\varepsilon^{\mathrm{M}} \\ &\mathscr{L}(k) \sim \varepsilon^{\mathrm{M}} k^{-L} && L = \alpha_{\mathrm{M}}^{\lambda'}(X_*) = \varepsilon^{\mathrm{M}}. \end{aligned} \tag{49}$$

In range III, the renormalised Lorentz force must be replaced by the external force $f(k, \omega)$ in equation (48a). Note that the absence of anomalous magnetic diffusion is specific of dimension three where $\alpha_{\mathrm{M}}^{\eta}(X, \kappa) = 0$: the magnetic field is then free; its energy spectrum reads

$$E^{\mathrm{M}}(k) \sim k^{1-\varepsilon^{\mathrm{M}}}. \tag{50}$$

The kinetic spectrum in ranges I and II is calculated in the appendix; we find

$$-1 < \varepsilon^{\mathrm{V}} < 3\varepsilon^{\mathrm{M}} \qquad E^{\mathrm{V}}(k) \sim (\varepsilon^{\mathrm{M}})^2 k^{1-\varepsilon^{\mathrm{M}}}. \tag{51}$$

The magnetic and kinetic spectra thus have the same exponent, but their intensities differ by a factor $\mathrm{O}(\varepsilon^{\mathrm{M}})^2$; in the ranges I and II, ε^{V} has no influence, the external force being non-relevant. However, in range III the kinetic spectrum results from a balance between the external force and the viscosity, renormalised by the small magnetic scales. It follows that for $3\varepsilon^{\mathrm{M}} \leqslant \varepsilon^{\mathrm{V}} < 4\varepsilon^{\mathrm{M}}$

$$E^{\mathrm{V}}(k) \sim k^{1-\varepsilon^{\mathrm{V}}+2\varepsilon^{\mathrm{M}}}; \tag{52}$$

the kinetic energy thus dominates the magnetic energy in the $k \to 0$ limit and nevertheless the transport coefficients in the large scales are determined by the magnetic small scales; again because of vertex renormalisation, the transition between the passive and magnetic regimes is not correctly predicted by comparing the kinetic and magnetic energy spectra (see figure 5).

We finally note that when the Lorentz force is relevant, the DIA or any consistent second order closure such as the EDQNM leads to

$$E^{\mathrm{V}}(k) \sim k^{1-\varepsilon^{\mathrm{M}}/2}, \tag{53}$$

a spectrum essentially prescribed by the non-local interactions. This result agrees with the prediction of the renormalisation group when the vertex renormalisation is arbitrarily dropped: the RG then yields a fixed point with $X_{**} = \frac{1}{2}X_*$ and $\alpha_{\mathrm{M}}^{\eta} = \varepsilon^{\mathrm{M}}$, stable if $\varepsilon^{\mathrm{V}} < 2\varepsilon^{\mathrm{M}}$. For $-1 < \varepsilon^{\mathrm{V}} < \frac{3}{2}\varepsilon^{\mathrm{M}}$, the spectrum given in (53) is recovered. When the external force dominates the Lorentz force ($\frac{3}{2}\varepsilon^{\mathrm{M}} < \varepsilon^{\mathrm{V}} < 2\varepsilon^{\mathrm{M}}$), the spectrum reads

$$E^{\mathrm{V}}(k) \sim k^{1-\varepsilon^{\mathrm{V}}+\varepsilon^{\mathrm{M}}}. \tag{54}$$

Note that in both ranges, the closure techniques predict a kinetic energy negligible when compared with the magnetic energy.

5.1.2. Model $A^{V}R^{M}$. This corresponds to $\varepsilon^{\mathrm{V}} = -1$ and $\varepsilon^{\mathrm{M}} > -1$. The kinetic driving may be renormalised. To first order in ε^{M}, the equation for $u = G/\kappa$ reads

$$\frac{\mathrm{d}u}{\mathrm{d}l} = \frac{1}{30\pi^2} u[-30\pi^2 - \tfrac{7}{2}u\kappa - 10u/(1+\kappa) - 8X + \tfrac{7}{2}X^2/u]. \tag{55}$$

The equation for X is the same as in model $\mathrm{R^V R^M}$. For negative ε^{M}, the passive regime is obtained; for positive ε^{M}, we look for a fixed point where u_* and κ_* vanish at the first order in ε^{M}; this is compatible with equation (55) provided $Y = X^2/u$ remains bounded. Indeed this ratio obeys the equation

$$\frac{\mathrm{d}Y}{\mathrm{d}l} = Y\left[30\pi^2 - \frac{4\kappa+34}{1+\kappa} u - \tfrac{7}{2}u\kappa + 4u^{1/2}Y^{1/2} - \tfrac{7}{2}Y\right] \tag{56}$$

for which

$$u_* = \mathrm{O}(\varepsilon^{\mathrm{M}2}) \qquad X_* = \mathrm{O}(\varepsilon^{\mathrm{M}}) \qquad Y_* = \tfrac{60}{7}\pi^2 \tag{57}$$

is a stable fixed point. The anomalous dimensions associated to the magnetic fixed point are the same as in model $\mathrm{R^V R^M}$, except that the small magnetic scales renormalise the kinetic forcing with an anomalous dimension

$$\alpha_{\mathrm{M}}^{D^{\mathrm{V}}} = \frac{7}{2} \frac{1}{30\pi^2} Y_* = 1.$$

The corresponding contribution to the kinetic energy spectrum is thus proportional to $k^{1+2\varepsilon^{\mathrm{M}}}$ and then negligible when compared with the Lorentz force contribution (proportional to $k^{1-\varepsilon^{\mathrm{M}}}$). The physics of the model $\mathrm{A^V A^M}$ is thus identical to that of the ranges I and II of the model $\mathrm{R^V R^M}$.

5.1.3. Models A^VA^M and R^VA^M. In this case $\varepsilon^M = -1$ and the fixed point associated to the passive regime is always reached (see equation (18*d*) with $d = 3$).

The competition between passive (trivial or kinetic) and magnetic regimes in dimension 3, for the different models, is summarised in figure 6.

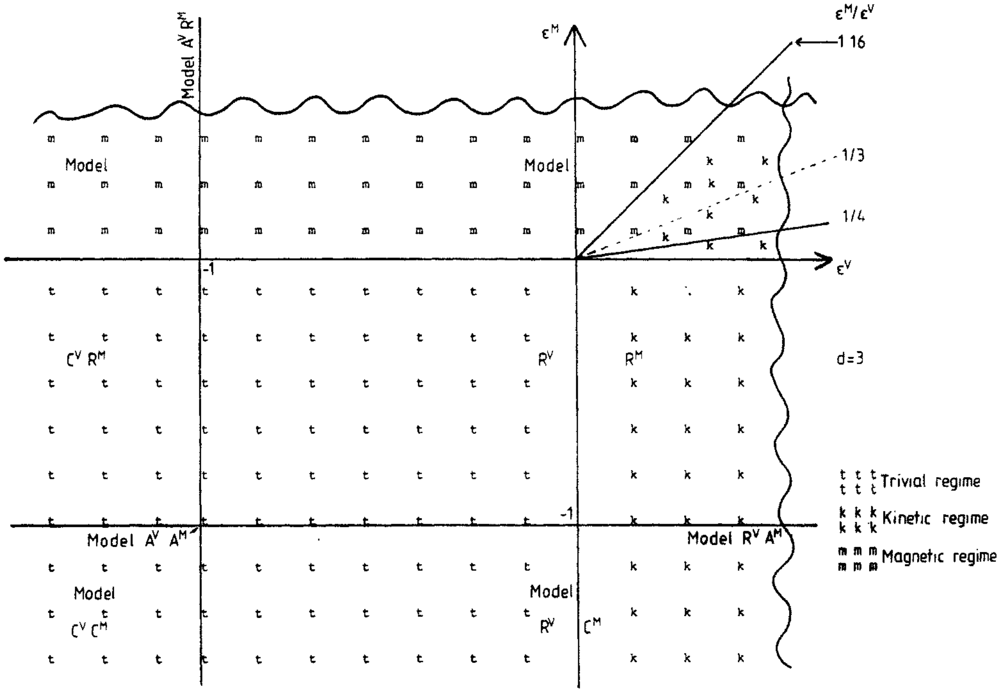


Figure 6. The different regimes obtained in dimension three according to the values of the crossover parameters ε^V and ε^M (defined by equations (5)). In the trivial regime, all the nonlinear effects are negligible. In the kinetic regime, the magnetic field behaves as a passive vector, but coefficients are renormalised by the kinetic small scales. In the magnetic regime renormalisation is due to the magnetic small scales. Note that in the range $\frac{1}{4} < \varepsilon^M/\varepsilon^V < \frac{1}{3}$, the Lorentz force is negligible even in the magnetic regime. The wavy line is a reminder of the validity limit of the RG calculation (when positive, ε^V and ε^M must be small).

5.2. Dimension $d = 2$

5.2.1. Model R^VR^M. This corresponds to small positive ε^V, and $\varepsilon^M > -2$; kinetic and magnetic driving are not renormalised. The equations for (u, κ, X) read:

$$\frac{\mathrm{d}u}{\mathrm{d}l} = \frac{u}{16\pi}[16\pi\varepsilon^V - \tfrac{1}{3}\kappa u - \tfrac{1}{2}u/(1+\kappa) - \tfrac{1}{4}X + \tfrac{1}{2}X/(1+\kappa)]$$

$$\frac{\mathrm{d}k}{\mathrm{d}l} = \frac{\kappa}{16\pi}[\tfrac{1}{2}u/(1+\kappa) - \tfrac{1}{8}u\kappa - \tfrac{1}{8}X - \tfrac{1}{2}X/(1+\kappa)] \qquad (58)$$

$$\frac{\mathrm{d}X}{\mathrm{d}l} = \frac{X}{16\pi}[16\pi\varepsilon^M + \tfrac{1}{8}X - \tfrac{1}{4}u - \tfrac{1}{8}u\kappa - u/(1+\kappa) + X/(1+\kappa)].$$

The point associated to the kinetic regime is the sole possibly stable fixed point of this system. The crucial element is the sign of the last term in the third equation which comes from the (negative) contribution of the magnetic small scales to the renormalised magnetic diffusivity. Even when the kinetic fixed point is linearly stable ($\varepsilon^{M} < 1.42\varepsilon^{V}$), this term can produce an indefinite growth of X. This is easily seen when considering the example of bare parameters κ_0 and X_0 which obey $(16\pi\varepsilon^{M} + (\kappa_0+9)X_0/(\kappa_0+1)) > 0$, with small $|\varepsilon^{M}|$ and $u_0 = 0$; in this case u remains zero and κ and X become infinite with l. In contrast, for finite negative ε^{M} and $X_0 < 32\pi|\varepsilon^{M}|/(18+|\varepsilon^{M}|)$, $\mathrm{d}X/\mathrm{d}l$ is negative and, for all u_0 and κ_0, X is exponentially driven to zero, leading to the kinetic regime.

This run-away has been numerically obtained for various bare parameters and different values of ε^{M}. When ε^{M} increases, the bassin of attraction of the fixed point associated to the kinetic regime decreases, and finally reduces for $\varepsilon^{M} = 1.427\ \varepsilon^{V}$ to the plane $X = 0$. When $\varepsilon^{M} > 1.427\ \varepsilon^{V}$, the MHD problem has no stable fixed point; even if small at scale O(1), the reduced nonlinear couplings grow indefinitely when $k \to 0$, making the RG procedure self-defeating. A relation may be conjectured between the failure of the RG approach and the possible existence of an inverse cascade of magnetic potential (Fyfe and Montgomery 1976, Pouquet 1978).

5.2.2. Model $A^{V}R^{M}(\varepsilon^{V} = 0; \varepsilon^{M} > -2)$. Recall that the fixed point $X = G = 0$, $\kappa = \kappa_*$, is stable for $\varepsilon^{M} < 0$ and unstable when $\varepsilon^{M} > 0$. The equations for κ and X are the same as in the model $R^{V}R^{M}$; G obeys

$$\frac{\mathrm{d}G}{\mathrm{d}l} = \frac{1}{8\pi} G[-G - \tfrac{3}{4}(\varepsilon^{M}+2)X + \tfrac{1}{2}X^2\kappa/G]. \tag{59}$$

The exponential decrease of X obtained previously with finite negative ε^{M} is still valid, and is consistent with an algebraic convergence of G towards zero; the last term of (59) is thus bounded, and the point associated to the passive regime is reached. There is no other stable fixed point, and for positive ε^{M} run-away always occurs.

5.2.3. Models $A^{V}A^{M}$ and $R^{V}A^{M}$. $\varepsilon^{M} = -2$ which ensures that the passive regime is reached.

The competition between passive regimes and run-away in dimension two is summarised in figure 7.

5.3. Non-integer dimensions

The strong difference between the results obtained in dimensions 2 and 3 when the Lorentz force is relevant, suggests the investigation of the case of non-integer dimensions; for simplicity we shall restrict ourselves to the model $R^{V}R^{M}(\varepsilon^{V} > 0$ and $\varepsilon^{M} > (2-d)$ for $d > 2$, or $\varepsilon^{V} > 0$ and $\varepsilon^{M} > -2$ for $d = 2)$. To deal with quantities which remain finite when $d \to \infty$, we introduce the rescaled variables $\tilde{u} = (2\pi)^d u/S_d$ and $\tilde{X} = (2\pi)^d X/S_d$. The recursion equations become

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}l} = \tilde{u}\left(\varepsilon^{V} - \frac{d^2-d-\varepsilon^{V}}{d(d+2)}\tilde{u}\kappa - \frac{d-1}{d}\frac{\tilde{u}}{1+\kappa} - \frac{d^2+d-4+\varepsilon^{M}}{d(d+2)}\tilde{X} - \frac{d-3}{d}\frac{\tilde{X}}{1+\kappa}\right)$$

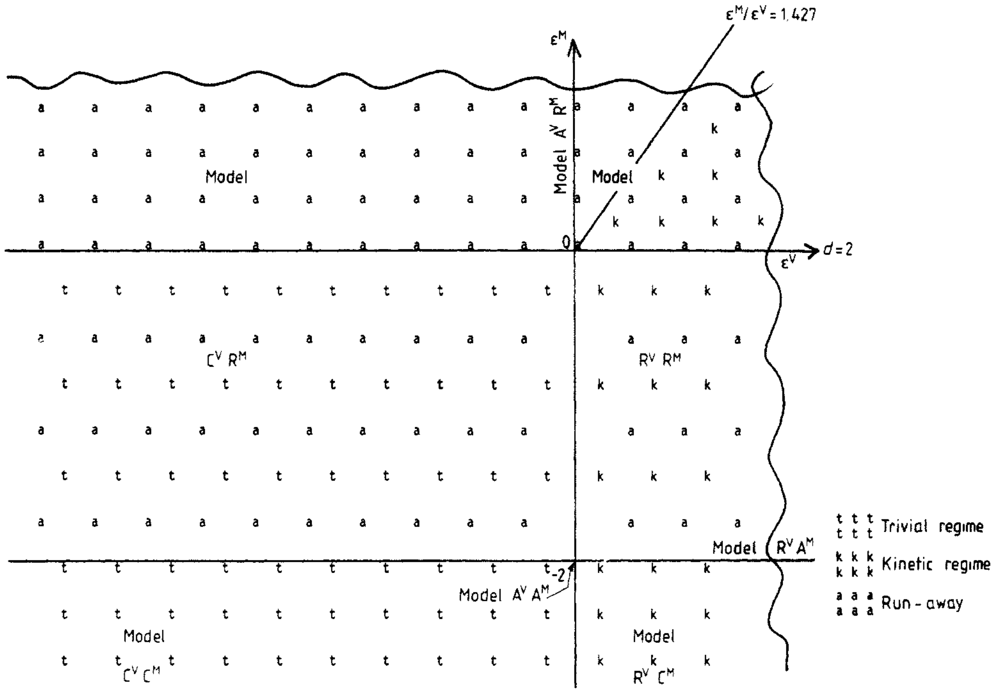


Figure 7. The different regimes in dimension two according to the values of the crossover parameters ε^{V} and ε^{M} (defined by equations (5)). Note that, even in presence of a stable regime, runaway of the figurative point may occur, making the RG self-defeating.

$$\frac{\mathrm{d}\kappa}{\mathrm{d}l}=\kappa\left(\frac{d-1}{d}\,\frac{\tilde{u}}{1+\kappa}-\frac{d^2-d-\varepsilon^{\mathrm{V}}}{2d(d+2)}\,\tilde{u}\kappa+\frac{d-3}{d}\,\frac{\tilde{X}}{1+\kappa}-\frac{d^2+d-4+\varepsilon^{\mathrm{M}}}{2d(d+2)}\,\tilde{X}\right) \tag{60}$$

$$\frac{\mathrm{d}X}{\mathrm{d}l}=X\left(\varepsilon^{\mathrm{M}}-\frac{2}{d(d+2)}-\tilde{u}-\frac{d^2-d-\varepsilon^{\mathrm{V}}}{2d(d+2)}\tilde{u}\kappa-2\frac{d-1}{d}\,\frac{\tilde{u}}{1+\kappa}\right.$$
$$\left.-\frac{d^2+d-8+\varepsilon^{\mathrm{M}}}{2d(d+2)}\tilde{X}-2\frac{d-3}{d}\,\frac{\tilde{X}}{1+\kappa}\right).$$

5.3.1. In dimensions close to 2, the only possibly stable fixed points are those associated with the passive regimes; divergence of the coupling constants occurs for certain bare values, while the passive fixed point is locally stable ($\varepsilon^{\mathrm{M}}\leqslant 0$); this run-away can be explained by the influence of the fixed point A:

$$A=\left(\tilde{u}_A=0,\ \kappa_A=0\,,\ \tilde{X}_A=\frac{2d(d+2)}{5d^2-3d-32}\,\varepsilon^{\mathrm{M}}\right) \tag{61}$$

which for negative ε^{M} has positive coordinates if $2\leqslant d<d_{\mathrm{c}}\simeq 2.848$ and is unstable in the direction $\tilde{u}=\kappa=0$. This point goes to infinity as d tends to d_{c}.

5.3.2. For $d>d_{\mathrm{c}}$, the point A has positive coordinates if ε^{M} is positive; for $d=3$, it yields the magnetic regime described in § 5.1.1. When linearised around this point the recursion equations read

$$\frac{\mathrm{d}}{\mathrm{d}l}\begin{bmatrix}\tilde{u}\\ \kappa\\ \tilde{X}-\tilde{X}_A\end{bmatrix}=\begin{bmatrix}\varepsilon^{\mathrm{V}}-\dfrac{4(d^2-5)\varepsilon^{\mathrm{M}}}{5d^2-3d-32} & 0\\ 0 & \dfrac{d^2-3d-8}{5d^2-3d-32}\varepsilon^{\mathrm{M}}\\ s & t\end{bmatrix}\begin{bmatrix}0 & \tilde{u}\\ 0 & \kappa\\ -\varepsilon^{\mathrm{M}} & \tilde{X}-\tilde{X}_A\end{bmatrix}. \tag{62}$$

The existence and stability conditions thus read

$$\begin{aligned}&2.8\simeq d_c<d<d_c'\simeq 4.701\\ &\varepsilon^{\mathrm{M}}>0\\ &\varepsilon^{\mathrm{V}}<\varepsilon^{\mathrm{M}}/a_A(d)\end{aligned} \tag{63}$$

where

$$a_A(d)=(5d^2-3d-32)/4(d^2-5). \tag{64}$$

When d increases from d_c to d_c', the stability of the fixed point A thus requires stronger conditions on the kinetic forcing: $a_A(d_c)=0$; $a_A(3)=\frac{1}{4}$; and $a_A(d_c')\simeq 0.941$.

The transport coefficients are renormalised according to

$$\begin{aligned}&\eta(k)\sim k^{-\Gamma} &&\text{with } \Gamma=(d-3)\frac{2(d+2)\varepsilon^{\mathrm{M}}}{5d^2-3d-32}\\ &\nu(k)\sim k^{-\Delta} &&\text{with } \Delta\frac{d^2+d-4}{5d^2-3d-32}\varepsilon^{\mathrm{M}}.\end{aligned} \tag{66}$$

The magnetic diffusive time is thus larger than the viscous time, as long as $2(d-3)(d+2)<d^2+d-4$, a condition insured by $d<d_c'$. When d approaches d_c', the viscous time catches up with the diffusive time. The renormalisation of the Lorentz force also depends on dimension:

$$\mathscr{L}(k)\propto\varepsilon^{\mathrm{M}}k^{-L} \tag{67}$$

with

$$L=\frac{4}{5d^2-3d-32}\varepsilon^{\mathrm{M}}. \tag{68}$$

5.3.3. The catching up of the viscous and diffusive times when d approaches d_c' suggests that we look for a magnetic fixed point B with a finite Prandtl number when $d>d_c'$. It reads

$$B=\left(\tilde{u}_B=0,\ \kappa_B=\frac{d^2-3d-8}{d^2+d-4},\ \tilde{X}_B=\frac{2d(d+2)}{3d^2+3d-16}\varepsilon^{\mathrm{M}}\right) \tag{69}$$

and has positive coordinates if ε^{M} is positive and $d>d_c'$. The points B and A coincide for $d = d_c'$. When linearised around the point B, the recursion equations become

$$\frac{\mathrm{d}}{\mathrm{d}l}\begin{bmatrix}\tilde{u}\\ \kappa-\kappa_B\\ \tilde{X}-\tilde{X}_B\end{bmatrix}=\begin{bmatrix}\varepsilon^{\mathrm{V}}-3\dfrac{(d^2-d-4)\varepsilon^{\mathrm{M}}}{3d^2+3d-16} & 0 & 0\\ t & -\dfrac{(d^2-3d-8)(d^2+d-4)\varepsilon^{\mathrm{M}}}{2(3d^2+3d-16)(d+2)(d-3)} & 0\\ t' & t'' & -\varepsilon^{\mathrm{M}}\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \kappa-\kappa_B\\ \tilde{X}-\tilde{X}_B\end{bmatrix}. \tag{70}$$

The stability condition is

$$\varepsilon^{\mathrm{V}}<\frac{1}{a_B(d)}\varepsilon^{\mathrm{M}} \tag{71}$$

with

$$a_B(d)=\frac{3d^2+3d-16}{3(d^2+d-4)}=1-\frac{4}{3(d^2+d-4)}; \tag{72}$$

$a_B(d)$ is equal to $a_A(d_c')\simeq 0.941$ for $d = d_c'$ and tends to unity when d goes to infinity.

In sufficiently high dimensions, a magnetic regime with finite Prandtl number can thus compete with the kinetic regime if $a_B(d)\leqslant a\leqslant a_{\mathrm{cr}}(d)$. The evolution of the fixed points A and B when d increases from $d_c\simeq 2.8$ to infinity is shown in figure 8.

5.3.4. We finally turn to the limit $d\to\infty$. The rescaled recursion equations (60) take the asymptotic form

$$\begin{aligned}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}l}&=\tilde{u}\left(\varepsilon^{\mathrm{V}}-\tilde{u}\frac{\kappa^2+\kappa+1}{\kappa+1}-\tilde{X}\frac{\kappa+2}{\kappa+1}\right)\\ \frac{\mathrm{d}\kappa}{\mathrm{d}l}&=-\frac{\kappa(\kappa-1)}{2(\kappa+1)}[\tilde{u}(\kappa+2)+\tilde{X}]\\ \frac{\mathrm{d}\tilde{X}}{\mathrm{d}l}&=\tilde{X}\left(\varepsilon^{\mathrm{M}}-\tilde{u}\frac{\kappa^2+\kappa+4}{2(\kappa+1)}-\tilde{X}\frac{\kappa+5}{2(\kappa+1)}\right).\end{aligned} \tag{73}$$

The fixed point corresponding to the trivial regime ($\tilde{X}=\tilde{u}=0$) is reached for negative ε^{V} and ε^{M}. The other possibly stable fixed points belong to the attractive plane $\kappa=1$:

(i) the kinetic fixed point ($\tilde{u}=\frac{2}{3}\varepsilon^{\mathrm{V}}$, $\kappa=1$, $\tilde{X}=0$) is the limit of the point studied in § 4. It is the only stable fixed point for $\varepsilon^{\mathrm{V}}\geqslant 0$, $\varepsilon^{\mathrm{M}}<\varepsilon^{\mathrm{V}}$;

(ii) the magnetic fixed point ($\tilde{u}=0$, $\kappa=1$, $\tilde{X}=\frac{2}{3}\varepsilon^{\mathrm{M}}$) is the limit of the point B. It is the only stable fixed point for $\varepsilon^{\mathrm{M}}\geqslant 0$, $\varepsilon^{\mathrm{V}}<\varepsilon^{\mathrm{M}}$;

(iii) finally for $\varepsilon^{\mathrm{V}}=\varepsilon^{\mathrm{M}}>0$, there appears a line of stable fixed points $\{\tilde{u}+\tilde{X}=\frac{2}{3}\varepsilon^{\mathrm{V}};\ \kappa=1\}$, which connects the kinetic and magnetic points (see figure 9).

As already mentioned, the Lorentz vertex corrections vanish when $d\to\infty$, and the transition from kinetic to magnetic regime is correctly predicted by the naive analysis ($a_{\mathrm{cr}}(\infty)=a_B(\infty)=1$). Except for the linear regime and for the border-line case $\varepsilon^{\mathrm{V}}=\varepsilon^{\mathrm{M}}>0$, universality is restored; the universal turbulent Prandtl number is always equal to unity.

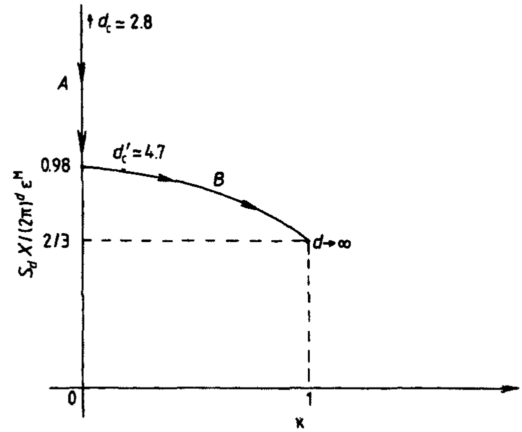


Figure 8. Evolution of the magnetic fixed point in the (κ, X) plane for the model $R^V R^M$, with the space dimension $d > d_c \simeq 2.8$, for small positive ε^M. X is rescaled by a factor $(1/\varepsilon^M) S_d/(2\pi)^d$. For $d > d_c'$, renormalised viscosity and diffusivity have the same scaling, leading to a universal finite Prandtl number.

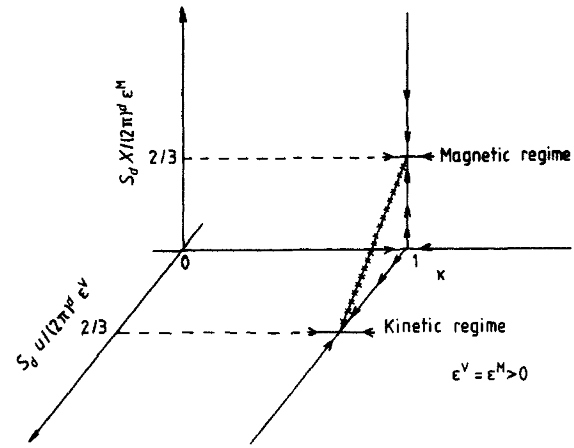


Figure 9. The stable non-trivial fixed points in the limit $d \to \infty$. For $\varepsilon^V = \varepsilon^M > 0$, a line of fixed point connects the kinetic and magnetic fixed points which exchange their stability (model $R^V R^M$). Note the universality of the Prandtl number which for all these fixed points is equal to unity.

6. Summary

Infrared properties of MHD turbulence, stirred by random forces and currents, have been studied using the renormalisation group. The recursion equations may be continued analytically to non-integer space dimensions but realisability constraints limit the dimension to $d \geq 2$. Different regimes are obtained according to space dimension, external drivings and in some cases fluid characteristics: a trivial regime where all the nonlinear couplings are negligible; a kinetic regime where the magnetic field behaves as a passive vector advected by turbulence; a magnetic regime where the

Lorentz force is manageably small and where the advection is negligible. Besides, in dimension 2 or close to 2, the renormalised coupling constants may diverge, which makes the RG procedure self-defeating.

The trivial and kinetic regimes exist in any space dimension. They correspond to a balance between kinetic and magnetic driving on the one hand, and viscous dissipation and magnetic diffusion on the other. In the kinetic regime, the transport coefficients, and possibly also the driving terms, are renormalised by the kinetic small scales. Turbulent viscosity and diffusivity then have the same scaling. In addition, the turbulent magnetic Prandtl number has a universal value which at the lowest order in ε^{V} depends on space dimension only and tends to 1 when $d \to \infty$.

The magnetic regime is found only for dimensions $d > d_c \simeq 2.8$. The effect of the kinetic small scales on the large scales is negligible and the renormalisation of the coupling (among them the Lorentz force) is only due to the magnetic small scales. (Possibly renormalised) magnetic diffusion and viscous dissipation balance respectively the magnetic driving and the Lorentz or the external force. The turbulent Prandtl number is infinite for $d_c < d < d_c' \simeq 4.7$, while for $d > d_c'$, it has a finite value which tends to 1 as $d \to \infty$.

No magnetic regime can be computed by the RG for $d < d_c \simeq 2.8$: in dimension $d < 3$, the contribution of the magnetic small scales to the turbulent diffusivity is negative and thus tends to destabilise the magnetic large scales. The kinetic and trivial regimes survive nevertheless but for a range of parameters which shrinks when the dimension decreases. In dimension close to two the electromotive force produces unbounded nonlinear effects in the large scales, no matter how small the Reynolds number at the reference scale. In dimension 2, similar behaviour is obtained for sufficiently large magnetic interaction parameter. This effect is possibly related to the infrared cascade of magnetic potential, conjectured for two-dimensional MHD by Fyfe and Montgomery (1976) on the basis of absolute equilibrium and by Pouquet (1978) from closure calculations. Such a situation is not within the scope of the renormalisation techniques which, for self-consistency, demand that the renormalised nonlinear couplings tend asymptotically either to zero or to a manageably small value.

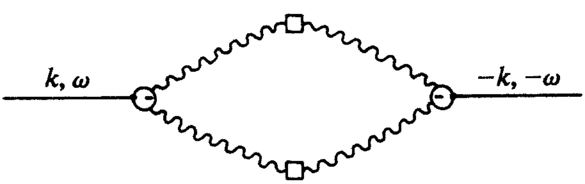
Acknowledgments

We are indebted to U Frisch and D Nelson for very helpful comments and suggestions.

Appendix

For space dimension $d = 3$, we go into the details of the calculation of the kinetic energy spectrum when the Lorentz force is relevant. The contribution of the Lorentz vertex renormalisation is printed in script. We have:

$$E^{\mathrm{V}}(k) \overset{k\to 0}{\sim} k^2 \int \mathrm{d}\omega$$



with

 $\propto X_* \ell^{-\varepsilon^M} P_{lmn}(k)$

$$—— = (-\mathrm{i}\omega + k^{2-\varepsilon^M} \ell^{-\varepsilon^M})^{-1}.$$

After integration over the internal frequencies, we obtain

$$E^V(k) \overset{k\to 0}{\sim} X_* k^2 \ell^{-\varepsilon^M} \int \frac{\mathrm{d}\omega}{\omega^2 + \nu^2 k^{4-2\varepsilon^M} \ell^{-2\varepsilon^M}} I(\boldsymbol{k}, \omega).$$

Here

$$I(\boldsymbol{k}, \omega) = \frac{D^{M2}}{2\eta} \int_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} \mathrm{d}^3 p \frac{k^2 a_{kpq} p^{1-\varepsilon^M} q^{1-\varepsilon^M} (p^2+q^2)}{p^2 q^2 [\omega^2 + \eta^2 (p^2+q^2)^2]}$$

and the coefficient a_{kpq} results from tensorial contraction. The change of variables $\boldsymbol{p} = |k| \boldsymbol{P}$, $\boldsymbol{q} = |k| \boldsymbol{Q}$, $\omega = k^2 \Omega$ yields

$$E^V(k) \overset{k\to 0}{\sim} X_*^2 k^{1-2\varepsilon^M} \ell^{-2\varepsilon^M} \int \frac{\mathrm{d}\Omega}{\Omega^2 + \nu^2 k^{-2\varepsilon^M} \ell^{-2\varepsilon^M}} J(\boldsymbol{k}, \Omega)$$

with

$$J(\boldsymbol{k}, \Omega) = \int_{\boldsymbol{P}+\boldsymbol{Q}=\boldsymbol{k}/|\boldsymbol{k}|} \mathrm{d}^3 P \frac{a_{1PQ} P^{1-\varepsilon^M} Q^{1-\varepsilon^M} (P^2+Q^2)}{P^2 Q^2 [\Omega^2 + \eta^2 (P^2+Q^2)^2]}.$$

One first integrates over Ω, which gives

$$E^V(k) \overset{k\to 0}{\sim} X_*^2 k \int \mathrm{d}^3 P \frac{a_{1PQ} P^{1-\varepsilon^M} Q^{1-\varepsilon^M}}{P^2 Q^2 [\nu + k^{\varepsilon^M} \ell^{\varepsilon^M} (P^2+Q^2)]}.$$

To evaluate this integral, we make another change of variables

$\boldsymbol{P} = k^{-\varepsilon^M/2} \ell^{-\varepsilon^M/2} \tilde{\boldsymbol{p}}$ $\quad$ $\boldsymbol{Q} = k^{-\varepsilon^M/2} \ell^{-\varepsilon^M/2} \tilde{\boldsymbol{q}}$ $\quad$ $\boldsymbol{k}/k = k^{-\varepsilon^M/2} \ell^{-\varepsilon^M/2} \tilde{\boldsymbol{k}}$

which gives

$$E^V(k) \overset{k\to 0}{\sim} X_*^2 k^{1-\varepsilon^M/2} \ell^{-\varepsilon^M/2} \int_{\tilde{p}+\tilde{q}=\tilde{k}} \mathrm{d}^3 p \frac{a_{\tilde{k}\tilde{p}\tilde{q}} \tilde{p}^{1-\varepsilon^M} \tilde{q}^{1-\varepsilon^M}}{\tilde{p}^2 \tilde{q}^2 [\nu + \eta (\tilde{p}^2 + \tilde{q}^2)]}.$$

When $k \to 0$, this last integral tends to

$$\int_{\mathbb{R}^3} \mathrm{d}^3 \tilde{p} \frac{a_{0\tilde{p}\tilde{q}} \tilde{p}^{2(1-\varepsilon^M)}}{\tilde{p}^4 [\nu + 2\eta \tilde{p}^2]}$$

which is finite.

References

de Dominicis C and Martin P C 1979 *Phys. Rev.* A **19** 419
de Dominicis C and Peliti L 1978 *Phys. Rev.* B **18** 353
Fisher M E 1974 *Rev. Mod. Phys.* **46** 597
Forster D, Nelson D R and Stephen M J 1976 *Phys. Rev. Lett.* **36** 867
—— 1977 *Phys. Rev.* A **16** 732
Fournier J-D 1977 *Thèse de 3ème Cycle* Université de Nice
Fournier J-D and Frisch U 1978 *Phys. Rev.* A **17** 747

Frisch U, Lesieur M and Sulem P-L 1976 *Phys. Rev. Lett.* **37** 895
Fyfe D and Montgomery D 1976 *J. Plasma Phys.* **16** 181
Garnier M, Alemany A, Sulem P-L and Pouquet A 1981 *J. Mécanique* **20** 233
Kraichnan R H 1959 *J. Fluid Mech.* **5** 497
—— 1971 *J. Fluid Mech.* **47** 513
Leslie D C 1973 *Developments in the Theory of Turbulence.* (Oxford: Clarendon)
Ma S K 1973 *Rev. Mod. Phys.* **45** 589
—— 1976 *Modern Theory of Critical Phenomena* (Reading, Mass.: Benjamin)
Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423
Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)
Orszag S A 1977 in *Fluid Dynamics, Les Houches Summer School of Physics* 1973 ed R Balian and J L Peube (New York: Gordon and Breach) p 235
Pouquet A 1978 *J. Fluid Mech.* **88** 1
Pouquet A, Fournier J-D and Sulem P-L 1978 *J. Physique Lett.* **39** 199
Pouquet A, Frisch U and Léorat J 1976 *J. Fluid Mech.* **77** 321
Pouquet A and Patterson G S 1978 *J. Fluid Mech.* **85** 305
Rose H A 1977 *J. Fluid Mech.* **81** 719
Rose H A and Sulem P-L 1978 *J. Physique* **39** 441
Sulem P-L, Fournier J-D and Pouquet A 1979 in *Dynamical Critical Phenomena and Related Topics, Lecture Notes in Physics* **104** (Berlin: Springer) p 321
Sulem P-L, Lesieur M and Frisch U 1975 *Ann. Géophys.* **31** 487
Wilson K G and Kogut J 1974 *Phys. Rep.* **12** 75